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Research Statement

My research interests center around **polynomial interpolation** in Algebraic Geometry and Commutative Algebra. The study of polynomial interpolation is deep-rooted in mathematics, reaching beyond Algebraic Geometry into Numerical Analysis, Computer Science, and others.

An initial encounter with polynomial interpolation might involve Lagrange interpolating polynomials. Given k pairs of real numbers (x_i, y_i) with x_i distinct and y_i arbitrary, there is always a unique polynomial $f \in \mathbb{R}[x]$ such that $f(x_i) = y_i$ for each $i = 1, 2, \ldots, k$ with $\deg f \leq k - 1$. A student with a background in calculus can easily verify that the polynomial

$$f(x) = \sum_{i=1}^{k} y_i \prod_{j=1, j\neq i}^{k} \frac{x - x_j}{x_i - x_j}$$

does precisely that.

The complexity of the interpolation problem increases significantly when the problem is posed in higher dimensions. In $\mathbb{R}^2$, the problem begins with a collection of k distinct points $(x_i, y_i) \in \mathbb{R}^2$ and arbitrary values z_i and we ask whether there is a polynomial $g(x, y)$ such that $g(x_i, y_i) = z_i$ for each $i = 1, 2, \ldots, k$. We may recontextualize the problem using linear algebra to give a new perspective.

As an example, consider three points $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) in $\mathbb{R}^2$. We may ask whether there is a linear polynomial $h(x, y) = ax + by + c$ such that $h(x_i, y_i) = 0$. This information may be described in the matrix

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

While the invertible matrix theorem gives conditions on whether the above system is consistent, observe that if such a solution (a, b, c) exists, it corresponds to a line with the equation $ax + by + c = 0$. For the 3 points to satisfy this one equation, that would imply that the 3 points would need to be collinear. On the other hand, if the 3 points were not collinear, then no such (a, b, c) exists that satisfies the system. This is a first example of how the position of the initial points matters greatly, unlike in the one-dimensional case where the distinct points x_i could be chosen arbitrarily.

The interpolation problem is similar to finding polynomials vanishing at points, and that is the starting point of Algebraic Geometry. One quickly encounters difficult interpolation problems, for example, given a collection of points $\{p_i = (x_i, y_i)\} \in \mathbb{R}^2$ with a multiplicity m_i assigned at each point p_i, it is an open question to determine the minimum value d for which there is a non-zero polynomial of degree d vanishing to order at least m_i at each p_i. If the points exhibit some symmetry or structure, this impacts what polynomials vanish at the points, as in the case of collinear points. That is, a *general* (or random) collection of points and a *special* collection of points will exhibit different spaces of vanishing polynomials. My research interest includes interpolation problem for special collections of points that are symmetric under some group action.

As these questions are within reach for undergraduate students and research in this area is readily accessible, I am looking forward to working with undergraduate students seeking research experience. A student who has taken Linear Algebra and Multi-variable Calculus would be prepared to consider these kinds of questions. The computer algebra systems SageMath and Macaulay2 are invaluable for learning, experimenting, and developing proofs [Sage, M2]. SageMath is a smooth introduction to coding mathematics as it familiarizes students with the Python programming language while seeing the mathematics they know come to life. Macaulay2 is a computer algebra software tailored for Commutative Algebra and Algebraic Geometry. These software have the advantage of grounding the fairly abstract field of Commutative Algebra, as well as providing an invitation for computer coding familiarity. Furthermore, it solidifies students as researchers who apply the scientific method, where they observe, conduct experiments with the aid of computers, and use these observations to lead them towards substantiating an answer to a question.

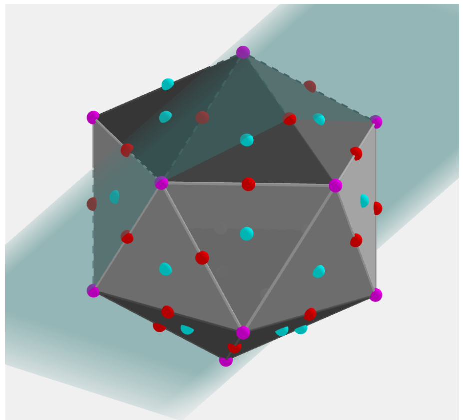

Figure 1: A marked icosahedron with one of its planes of symmetry

1 Current Work

Waldschmidt Constants

First, we would like our collection of points to be in $\mathbb{P}^2$, with coordinate ring $S = \mathbb{C}[x, y, z]$. For a point $p \in \mathbb{P}^2$, there is a maximal ideal $I_p \subseteq S$ corresponding to the curves that pass through p. If $\{p_i\}$ is a collection of distinct points in $\mathbb{P}^2$, there is a corresponding ideal I of the points obtained by intersecting I_{p_i}. More generally for a positive integer m, define the *m-th symbolic power* of an ideal I to be

$$I^{(m)} = \bigcap_i I_{p_i}^m$$

The ideal $I^{(m)}$ geometrically corresponds to the curves in S that pass through each point at least m times. Algebraically, each such $f \in I^{(m)}$ vanishes at each p_i and so do all the $(m-1)$-th order partial derivatives of f. For an ideal I, denote $\alpha(I)$ to be the least positive degree of an element in I.

Let I be the defining ideal of n points in $\mathbb{P}^2$. We define the Waldschmidt constant by

$$\widehat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}.$$

These were first defined in [Wal77] as the reciprocal of a multi-point Seshadri constant. The Waldschmidt constant measures the speciality of a collection of points. The most special a collection of n points could be if all the points are collinear, resulting in a Waldschmidt constant of 1. Therefore 1 is the minimum value a Waldschmidt constant can take, while on the other hand $\widehat{\alpha}(I) \leq \sqrt{n}$ is always true. A conjecture of Nagata [Nag59] predicts the Waldschmidt constant for a general collection of points.

Conjecture 1.1 (Nagata)**.** *Let I be the ideal defined by a general collection of $n \geq 9$ points in $\mathbb{P}^2$. Then $\widehat{\alpha}(I) = \sqrt{n}$.*

The Icosahedral Line Configuration

The icosahedron has 12 vertices, 20 faces, and 30 edges. We color the icosahedron by magenta for each vertex, blue for the center of each face, and red for the midpoint of each edge. In addition, the icosahedron has 15 planes of symmetry, where reflection across one of these planes leaves the icosahedron invariant. See Figure 1. Projectivizing the mirror planes of the icosahedron produces the icosahedral line configuration $\mathcal{A}$ with the polynomial equation $\phi_{15} = 0$, obtained by multiplying the equations of the 15 lines. The configuration $\mathcal{A}$ (see Figure 2) consists of 15 lines with 31 points of intersection 15 double points, 10 triple points, and 6 quintuple points.

Let $I_{\mathcal{A}}$ be the ideal in S of these 31 points. By construction, the 31 points are already in a special position. To consider how special the points are, we compute the Waldschmidt constant of $I_{\mathcal{A}}$.

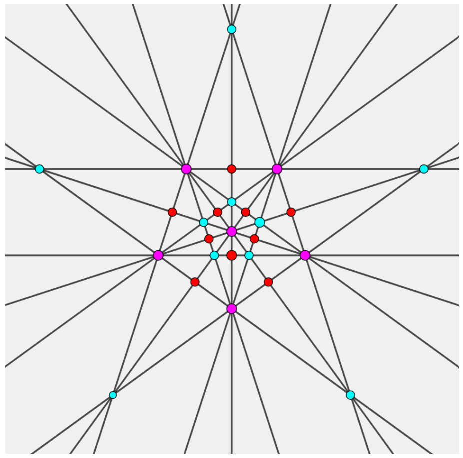

Figure 2: The icosahedral line configuration $\mathcal{A}$. Note there are five double points on the line at infinity.

Theorem 1.2. *Let $I_{\mathcal{A}}$ be the defining ideal of the* 31 *points of the icosahedral line configuration $\mathcal{A}$. Then*

$$\widehat{\alpha}(I_{\mathcal{A}}) = \frac{11}{2}.$$

According to Conjecture 1.1, a general collection of 31 points would correspond to a Waldschmidt constant of $\sqrt{31} \approx 5.567$. From a general arrangement of 31 points, we specialize the points according to the incidences of $\mathcal{A}$. Arranging the points in such a specific manner surprisingly only decreases the Waldschmidt constant by approximately 0.0677.

A key step in the computation of the Waldschmidt constant of $I_{\mathcal{A}}$ was the construction of curves in $\mathbb{P}^2$ having high multiplicities at the 31 points of $\mathcal{A}$. The line configuration, given by $\phi_{15} = 0$, is already a great example of such a curve. However, the Waldschmidt constant suggests to consider curves that have the same multiplicity at each point, which $\phi_{15} = 0$ does not. The search for such curves is aided tremendously by exploiting the symmetry group G of the icosahedron.

The ring of invariant polynomials under the action of G is generated by finitely homogeneous polynomials ϕ_2, ϕ_6 and ϕ_{10} of degree 2, 6 and 10 [DK15]. These can be found geometrically (see also Figure 3):

- $\phi_2 = x^2 + y^2 + z^2$ preserves distance.
- There are six pairs of opposite vertices, each sandwiching a plane that slices the icosahedron in half. Take the product of the six polynomial equations of each plane to obtain ϕ_6.
- Similarly, there are 10 pairs of centers of opposite faces, each pair having a plane that slices the icosahedron in half. Taking the product of the ten polynomial equations of each plane yields ϕ_{10}.

The invariant polynomials ϕ_6 and ϕ_{10} already pass through some of the points of $\mathcal{A}$. Those polynomials along with ϕ_2 were used to generate more G-invariant polynomials with incidence properties with the points of $\mathcal{A}$. For example, we were able to obtain a curve given by a polynomial ϕ_{30} of degree 30 with multiplicities $2, 6$, and 6 at the quintuple, triple and double points of $\mathcal{A}$ respectively. The curves cut out by the polynomials ϕ_6, ϕ_{15}, and ϕ_{30} were then used to construct a family of curves of degree $55k + 2$ with a multiplicity of $10k$ at each of the 31 points of $\mathcal{A}$. The G-invariance provided an indispensable tool to search for and study these curves. The polynomial ϕ_{30} is a sum of 118 monomials with large coefficients; the first three terms are expressed below

$$\phi_{30} = (1913625\omega - 3098250)x^{24}y^6 + (1458000\omega - 2278125)x^{22}y^8 + (-58320000\omega + 92947500)x^{20}y^{10} + \cdots$$

where ω is the golden ratio. However, expressing ϕ_{30} in terms of G-invariant polynomials has only 4 terms with the coefficients either being 1 or 2.

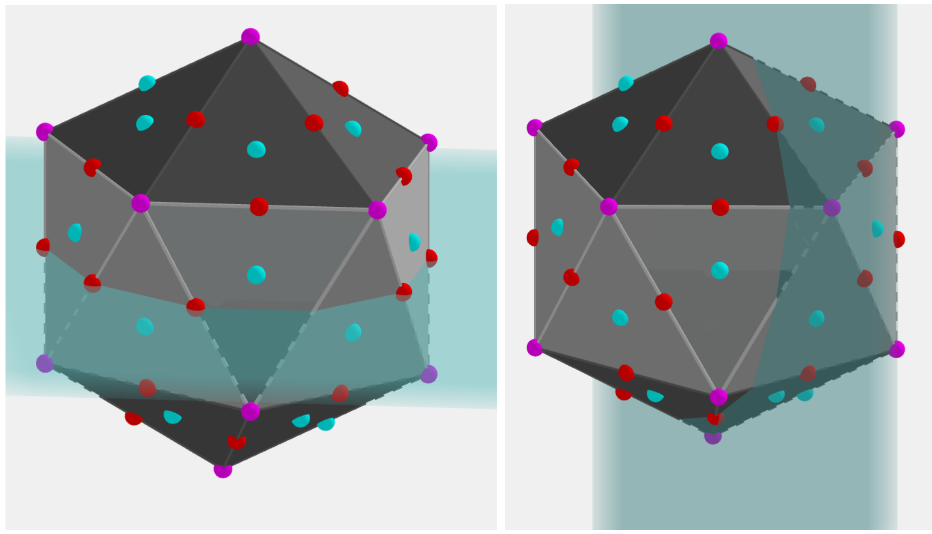

Figure 3: The icosahedron with a plane sandwiched between two opposite vertices (left) and a plane sandwiched between two opposite centers (right).

Containment Problem

While it is hard to separate Algebraic Geometry from its Commutative Algebra foundations, there are fruitful questions to explore if one remains in the field of Commutative Algebra. The Containment Problem may serve as a bridge between the two fields but it very much is an algebraic question. The containment problem is straightforward: for what integers $m, r \geq 1$ does the containment

$$I^{(m)} \subseteq I^r$$

hold? This question has been studied extensively for I is an ideal defined by a set of points in $\mathbb{P}^2$. The authors in [ELS01] showed that $I^{(m)} \subset I^r$ for $m/r \geq 2$. This bound is sharp for a general I and was proven in [BH10]. However, for the special configuration of points given by the Klein and Wiman line configurations, the authors proved both ideals satisfied $I^{(3)} \subseteq I^2$ [BDRH19]. To see whether our special collection of points given by the icosahedral line configuration $\mathcal{A}$ yields a similar result, we studied the *resurgence* of $I_{\mathcal{A}}$. Introduced by Bocci-Harbourne, the resurgence of a general I is defined to be

$$\rho(I) = \sup\left\{\frac{m}{r} \mid I^{(m)} \not\subseteq I^r\right\}.$$

It is often the case that an asymptotic invariant is sometimes initially easier to handle. Indeed by [ELS01, HoHu], it is known that $I^{(tr)} \subseteq I^r$ for $t \gg 0$. To track this, the *asymptotic resurgence* is defined to be

$$\widehat{\rho}(I) = \sup\left\{\frac{m}{r} | I^{(mt)} \not\subseteq I^{rt} \text{ for } t \gg 0\right\}.$$

We recently were able, by computer experimentation, to prove the following:

Theorem 1.3. *Let $I_{\mathcal{A}}$ be the defining ideal of the* 31 *points of the icosahedral line configuration $\mathcal{A}$. Then*

$$I^{(9)} \not\subseteq I^8.$$

In particular, $\rho(I_{\mathcal{A}}) = 9/8$ and $\widehat{\rho}(I_{\mathcal{A}}) = 12/11$.

2 Future Work

Hyperplane Arrangements

Similar questions were posted in [BDRH19], where their analysis was done for the Klein line configuration $\mathcal{K}$ and the Wiman line configuration $\mathcal{W}$. What makes the configuration of points $\mathcal{L} = \mathcal{K}, \mathcal{W}, \mathcal{A}$ similar is that each has a group of symmetries naturally associated to the configuration and this group is a *complex reflection group*. These groups constitute a nice playground of examples of groups that naturally yield a hyperplane arrangement. In the case of the three arrangements above, the complex reflection groups are rank 3 groups, indicating that the (projective) hyperplane arrangement they yield lies in $\mathbb{P}^2$. In general, a complex reflection group of rank n will admit a hyperplane arrangement in $\mathbb{P}^{n-1}$. Due to [ST54], there is a finite classification of exceptional complex reflection groups. The rank 3 complex reflection groups have been studied in this context [Cal22, BDRH19].

The natural next step is to investigate the rank 4 complex reflection groups. The rank 4 reflection group G_{28} yields a plane configuration in $\mathbb{P}^3$ consisting of the following combinatorial data: 24 planes intersecting at 54 lines, 36 of which are triple and 18 are quadruple, and 120 points, 96 of which are quadruple and 24 are nonuple.

From the algebraic perspective, the reduced singular locus of the arrangement is $I_{\mathcal{B}} = \sqrt{\mathrm{Jac}(\Pi\, l_H)}$ where l_H is a defining equation of the 24 hyperplanes [DS21]. Alternately, one can still consider the ideal $J_{\mathcal{B}}$ defined by the 120 points of intersection. Both of these ideals are geometrically interesting but new methods would need to be found and utilized to compute $\widehat{\alpha}(I_{\mathcal{B}})$ or $\widehat{\alpha}(J_{\mathcal{B}})$.

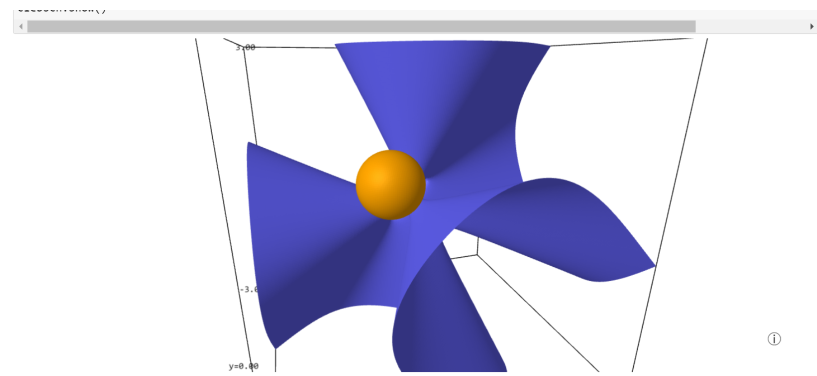

Figure 4: The Clebsch cubic surface and a sphere intersecting at 4 tangency points.

Coble Surfaces

Another example of an exotic configuration of points is the nodes of a Coble curve C: a rational sextic curve with 10 nodes. These were studied extensively by Arthur Coble in the early 20th century in his seminal paper [Cob19]. For a set of 10 general points, there does not exist a sextic curve passing through the 10 points with multiplicity 2. The existence of such a curve requires that the 10 points be quite special. If such a sextic is given, the blow-up S of $\mathbb{P}^2$ at the 10 nodes satisfies $|-K_S| = \varnothing$ and $|-2K_S| \neq \varnothing$ where K_S is the canonical bundle of S. We say that S is a Coble surface.

Coble asked [Cob19] whether given a Coble Surface S and its Coble curve C, does the restriction of the automorphisms of S to automorphisms of C induce an injection $\mathrm{Aut}(S) \to \mathrm{Aut}(C) \cong \mathrm{PGL}_2(\mathbb{C})$? I have considered the case for when S is the blow-up of $\mathbb{P}^2$ at the 4 tangency points of the Clebsch cubic surface and a sphere centered on the Clebsch's rotational axis (See Figure 4). Since the Clebsch surface is a blow-up of $\mathbb{P}^2$ at 6 points, blowing-up at 4 additional points does indeed result in a Coble surface S. The group $\mathrm{Aut}(S)$ is generated by Bertini involutions [DV06]. To attempt to answer Coble's question for this specific Coble Surface S, we would need to obtain the 45 Bertini Involutions associated to S. Having done so, it should be straightforward to search for elements in the kernel of the map $\mathrm{Aut}(S) \to \mathrm{PGL}_1(\mathbb{C})$.

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