PROJECTIVE MONOID SCHEMES AND EXTENDED FANS

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Contents

1. Introduction

Toric geometry studies a contemporary collection of combinatorially defined objects in algebraic geometry called toric varieties, first formally defined in the 1970's [CLS11]. In toric geometry, it is a foundational theorem that the categories of affine normal toric varieties, convex rational polyhedral cones and *toric* monoids are all equivalent. Recently, this was partially extended to an equivalence between the categories of *pointed toric* monoids and *extended* convex rational polyhedral cones [HMU19]. In this paper, we provide examples that extend these results further using *monoid schemes* (or $\mathbb{F}_1$-scheme in some of the literature) from discrete geometry as a construction to glue pointed toric monoids. In particular, we demonstrate a bijection between morphism sets between non-affine monoid schemes and morphism sets of extended *fans*. This is a first step towards potential future work that we anticipate to be true: the category of extended convex rational polyhedral fans is equivalent to the category of toric monoid schemes (i.e. monoid schemes glued affine locally using toric pointed monoids).

2. Background

First we discuss preliminary ideas to lay down the foundations needed to discuss locally monoidal spaces and extended fans in Section 3. The following is excerpted from [HMU19].

2.1. **Monoids.** If a set M satisfies closure under an associative operation and has an identity element, we call M a *monoid*. A a common example is the natural numbers, denoted $\mathbb{N}$, which we assume throughout to include it's identity under addition 0.

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An *ideal* of a monoid M is a (potentially empty) subset $N \subset M$ that inherits the operation of M and observes an absorbing structure. By this we mean for all $n \in N$ and $m \in M$, $m \cdot n \in N$ and $n \cdot m \in N$. An example of an ideal in $\mathbb{N}$ is the subset $\mathbb{N}^+$.

We call a proper ideal p *prime* in $\mathbb{M}$ if for $m \cdot n \in p$, then $m \in p$ or $n \in p$. The *prime spectrum* of a monoid is the set of all prime ideals in M.

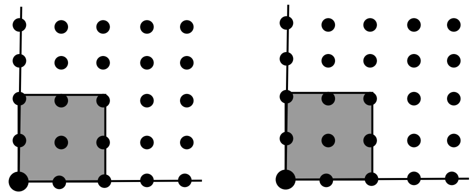
For a monoid M, it is not required for elements to have inverses in M. For example, in $\mathbb{N}$, 1 does not have an inverse. To fix this, we can invert the elements of any subset that is closed under addition. Let $N \subset M$ be a set that contains the identity and is closed under addition. Then $N^{-1}M = M\left[\frac{1}{n}\right]$ for $n \in N$. We call this *localizing at a set*. Following the nomenclature of commutative algebra, when we say that we localize a monoid M at a prime ideal $p \subset M$, we invert the elements that are not in p. So the *localization of M at p* is $M[\frac{1}{M-p}]$.

Our focus will be on *toric monoids*; that is, monoids that are finitely generated, saturated, cancellative and are torsion-free (see [HMU19, Section 2] or [CnHWW15, Section 2] for standard background definitions in monoid theory). Toric monoids are equivalently defined as monoids that embed as a saturated subset of a lattice or, also equivalently, as subsets of a lattice produce by intersecting a lattice with a cone (defined below). The natural numbers $\mathbb{N}^k$ for k a positive integer are an example of toric monoids. In fact, they and their localizations will be the only kind of toric monoids discussed in this paper.

2.2. **Cones.** To discuss cones, we must first introduce lattices. A *lattice* N is a free abelian group of finite rank. A lattice of rank n is isomorphic to $\mathbb{Z}^n$. The *dual lattice* M of N is defined as the lattice of the dual induced $\mathbb{R}$-vector space, that is, $M = \mathrm{Hom}(N, \mathbb{Z})$. Since both N, M are isomorphic to $\mathbb{Z}^k$, we may treat the inner product of $n \in N$ and $m \in M$ as the standard inner product on $\mathbb{R}^k$. The *realification* of N (i.e. the induced $\mathbb{R}$-vector space) is given by N tensor product $\mathbb{R}$ over the integers, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We now can define half spaces as $H_i = \{u \in N_{\mathbb{R}} \mid \langle u, v_i \rangle \geq 0\}$ for $v_i \in M$ using the standard inner product.

Finally, we define a *strictly convex rational polyhedral cone* σ (or just cone for short) to be a pair (σ, N) where N is a lattice and σ is a finite intersection of half-spaces H_i in $N_{\mathbb{R}}$ where each $v_i \in N^\vee = M$. The *dual cone* to $\sigma, \sigma^\vee = \{v \in M_{\mathbb{R}} \mid \langle u, v \rangle$ for all $u \in \sigma\}$, is a cone in $M_{\mathbb{R}}$. A *face* of a cone is $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee$. To denote that τ is a face of a cone σ, we write $\tau \preceq \sigma$. The *dual face* to τ is a face of $\sigma^\vee$ given by $\tau^* = \sigma^\vee \cap \tau^\perp$ where $\tau^\perp = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle = 0$ for all $v \in \tau\}$.

A *morphism* of cones $f : (\sigma, N) \to (\sigma', N')$ is a map in $\mathrm{Hom}(N, N')$ extended to a linear transformation of vector spaces $N_{\mathbb{R}} \to N'_{\mathbb{R}}$ so that $f(\sigma) \subseteq \sigma'$. A basic fact of toric geometry is that every cone (σ, N) can be realized as $\sigma_S = \mathrm{Hom}(S, \mathbb{R}_{\geq 0})$ where $S = \sigma^\vee \cap M$ is a toric monoid and the Hom is taken in the category of monoids. For example, the cone $\{(x, y) \mid x, y \in \mathbb{R}_{\geq 0}\}$, the first quadrant in the plane, can be realized as $\sigma = \mathrm{Hom}(\mathbb{N}^2, \mathbb{R}_{\geq 0})$ along with the lattice $N = \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$.



$\sigma = Hom\,(\mathbb{N}^2, \mathbb{R}_{\geq 0})$ $\qquad\qquad$ $\sigma^\vee$

FIGURE 1. $\sigma = \mathrm{Hom}(\mathbb{N}^2, \mathbb{R}_{\geq 0})$ and $\sigma^\vee$

We see that σ is generated by two elements, $e_1 = (1, 0)$ and $e_2 = (0, 1)$ and thus has two faces.

We will always write σ_M for the cone realized as $\mathrm{Hom}(M, \mathbb{R}_{\geq 0})$ where, again, the Hom is taken in the category of monoids.

2.3. **Category Theory.** A category $\mathcal{C}$ consists of a collection of objects and a collection of morphisms between those objects such that the following axioms are held:

(1) For every $A \in \mathcal{C}$, there exists the identity morphism on $A, Id_A : A \to A$
(2) For $A, B, C \in \mathcal{C}$, if $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, then $g \circ f \in \mathrm{Hom}(A, C)$
(3) For morphisms f, g, h, composition is associative, $f \circ (g \circ h) = (f \circ g) \circ h$

The notation $\mathrm{Hom}(A, B)$ denotes the set of morphisms from objects A to B in category $\mathcal{C}$. To avoid confusion when discussing different categories, we will specify the category for a Hom set, $\mathrm{Hom}_{\mathcal{C}}(A, B)$. For categories $\mathcal{C}, \mathcal{D}$, a *contravariant functor* F takes an object $C \in \mathcal{C}$ and sends it to an object $F(C) \in \mathcal{D}$ and a morphism in $f : C_1 \to C_2$ in $\mathcal{C}$ to the morphism $F(f) : F(C_2) \to F(C_1)$ in $\mathcal{D}$.

Some examples of categories are **TorMon**, whose objects are toric monoids and morphisms are identity-preserving homomorphisms, and **RPC**, whose objects are rational polyhedral cones and morphisms are cone morphisms.

3. Extended Cones and Fans

3.1. **Extended Cones.** Recall for a toric monoid M, any cone is realized as $\sigma = \mathrm{Hom}(M, \mathbb{R}_{\geq 0})$. An *extended rational polyhedral cone* is produced from any cone by taking the compactification $\overline{\sigma} = \mathrm{Hom}(M, \overline{\mathbb{R}}_{\geq 0})$, where $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$. This is a compact topological space containing σ as a dense open subset. Away from σ, this construction introduces *faces at infinity* $F(\tau, \tau')$, the set of points in $\overline{\tau}$ that are in the τ' direction to infinity. Our focus will be those faces at infinity of the form $F(\sigma, \tau)$. The details in this case are as follows. Given a face τ of σ let N_τ denote the sublattice of N spanned by the points in $\tau \cap N$ and set $N(\tau) = N/N_\tau$. The cone quotient σ/τ is a cone in $N(\tau)_{\mathbb{R}}$ described by the image of all the faces of σ containing τ through the induced quotient map of vector spaces $N_{\mathbb{R}} \longrightarrow N(\tau)_{\mathbb{R}}$:

$$F(\sigma, \tau) \cong \sigma/\tau = \{[\tau'] \subset N(\tau)_{\mathbb{R}} | \tau \preceq \tau' \preceq \sigma\}.$$

For a better understanding of how this works, lets discuss the extended cone $\overline{\sigma} = \mathrm{Hom}(\mathbb{N}^2, \overline{\mathbb{R}}_{\geq 0})$ produced from our example above.

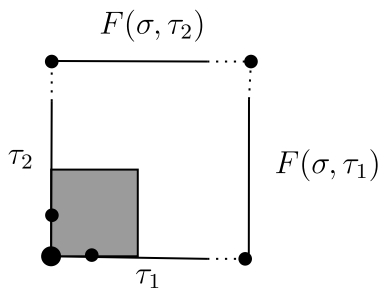


FIGURE 2. $\overline{\sigma} = \mathrm{Hom}(\mathbb{N}^2, \overline{\mathbb{R}}_{\geq 0})$

The face at infinity $F(\sigma, \tau_1)$ is the set of points such that e_1 is sent to ∞ and e_2 is sent to a real number. The notation $F(\sigma, \tau_1)$ denotes the face at infinity of $\overline{\sigma}$ given by taking a quotient of σ along τ_1 and gluing in the resulting quotient space at infinity. This open face not include the point (∞, ∞), whereas the *extended* face at infinity $\overline{F}(\sigma, \tau_1)$ does. The full extended cone is depicted in Figure 2.

A *morphism* of extended rational polyhedral cones is a continuous map $f : \overline{\sigma}_1 \to \overline{\sigma}_2$ so that the restriction $f|_{\sigma_1}$ of f to the interior cone is a cone morphism. With this we recall from [HMU19, Section 2] an important way to decompose any morphism of extended rational polyhedral cones.

Lemma 3.1 (Structure Lemma)**.** *Any morphism $f : \overline{\sigma}_1 \to \overline{\sigma}_2$ decomposes uniquely into a factorization $\alpha : \overline{\sigma}_1 \to \overline{F}(\sigma_2, \tau)$ for some τ a face of σ_2 where $\alpha|_{\sigma_1}$ is induced from a morphism of cones (i.e. the image meets the open interior of $\overline{F}(\sigma_2, \tau)$), and i is the inclusion of $\overline{F}(\sigma_2, \tau)$ into $\overline{\sigma}_2$ as an extended face at infinity.*

For a thorough introduction to extended cones, see [HMU19, Section 2].

3.2. **Extended Fans.** A *rational polyhedral fan* (or fan for short) is a finite collection of cones Δ all under the same lattice N and is closed as a collection under taking intersections of cones. By this, we mean that if we intersect two cones such that the intersection is nonempty, then the resulting intersection is itself a cone. A *morphism* of fans $(\Delta, N) \to (\Delta', N')$ is a morphism $f \in \mathrm{Hom}(N, N')$ of lattices who's canonical extension to the linear map of vector spaces $f_{\mathbb{R}} : N_{\mathbb{R}} \to N'_{\mathbb{R}}$ restricts to a morphism of cones for each $\sigma \in \Delta$.

$$\begin{tikzcd}
\overline{\sigma}_1 \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & \overline{\sigma}_2 \\
& \overline{F}(\sigma_2, \tau) \arrow[ur, "i"'] &
\end{tikzcd}$$

FIGURE 3. Structure Lemma

As above, given a cone σ of Δ let N_σ denote the sublattice of N spanned by the points in $\sigma \cap N$ and set $N(\sigma) = N/N_\sigma$. Following notation in toric geometry, we denote by $\mathrm{Star}(\Delta, \sigma)$ the fan in $N(\sigma)_\mathbb{R}$ given by the image of all cones $\sigma' \in \Delta$ containing σ as a face through the induced quotient map of vector spaces $N_\mathbb{R} \longrightarrow N(\sigma)_\mathbb{R}$:

$$\mathrm{Star}(\Sigma, \sigma) = \{[\sigma'] \subset N(\sigma)_\mathbb{R} | \sigma \preceq \sigma' \preceq \Delta\}.$$

The cone quotients $\{\sigma'/\sigma | \sigma \preceq \sigma'\}$ form the cones of $\mathrm{Star}(\Delta, \sigma)$, while σ'/σ and σ''/σ meet in a face τ/σ precisely when $\sigma \preceq \tau \preceq \sigma'$ and $\sigma \preceq \tau \preceq \sigma''$.

For a given fan Δ, the associated *extended rational polyhedral fan* $\overline{\Delta}$ can be produced by extending all cones in Δ and gluing along extended common faces. This induces gluing of the extended cones

$$\overline{\sigma} = \bigsqcup_{\tau \preceq \sigma} F(\sigma, \tau)$$

among all the ended cones in $\overline{\Delta}$, resulting in

$$\overline{\Delta} = \bigsqcup_{\sigma \in \Delta} \mathrm{Star}(\Delta, \sigma).$$

We denote by $\overline{\mathrm{Star}}(\Delta, \sigma)$ the extended fan of one of these maximal fans at infinity. By the same construction, we have

$$\overline{\mathrm{Star}}(\Delta, \sigma) = \bigsqcup_{\sigma' \in \Delta, \sigma \preceq \sigma'} \mathrm{Star}(\Delta, \sigma').$$

A morphism of extended fans is a continuous map of topological spaces $\overline{f} : \overline{\Delta} \to \overline{\Delta'}$ such that the restriction $\overline{f}$ to any extended cone $\overline{\sigma} \in \overline{\Delta}$ factors through a morphism of extended cones.

Let $f : \overline{\Delta} \to \overline{\Delta'}$ be a morphism of extended fans. Applying Lemma 3.1 cone by cone, one obtains an analogous structure lemma about unique factorization of morphsims of extended fans: there is a unique cone σ' of $\overline{\Delta'}$ so that the map f factors first as a map $\overline{f}$ that sends $\overline{\Delta}$ to $\overline{\mathrm{Star}}(\Delta', \sigma')$ and whose image meets the open interior of $\overline{\mathrm{Star}}(\Delta', \sigma')$, and then by i the inclusion of $\overline{\mathrm{Star}}(\Delta', \sigma')$ into $\overline{\Delta'}$ as a fan at infinity.

$$\begin{tikzcd}
\overline{\Delta} \arrow[rr, "f"] \arrow[dr, "\overline{f}"'] & & \overline{\Delta'} \\
& \overline{\mathrm{Star}}(\Delta', \sigma') \arrow[ur, "i"'] &
\end{tikzcd}$$

FIGURE 4. Decomposition of a morphism of fans

It is always best to think of extended fans as gluings of extended cones along faces. For example, consider the two fans we will be using often, $\overline{\Delta}_1$ and $\overline{\Delta}_2$, as shown below in Figure 5. The extended fan $\overline{\Delta}_1$ is two copies of the extended cone $\overline{\sigma}_{\mathbb{N}^\infty} = \mathrm{Hom}(\mathbb{N}^\infty, \overline{\mathbb{R}}_{\geq 0})$ glued along the origin (the origin itself is a zero-dimensional vector space, which agrees with the idea that the intersection of these two cones is a single point). The extended fan $\overline{\Delta}_2$ is three copies of the extended cone $\overline{\sigma}_{(\mathbb{N}^2)^\infty}$ glued along appropriate one-dimensional faces. Further in the paper, we will refer to $\overline{\Delta}_1$ as $\overline{\Delta}_{\mathbb{P}^1_{\mathbb{F}1}}$ and $\overline{\Delta}_2$ as $\overline{\Delta}_{\mathbb{P}^2_{\mathbb{F}1}}$.

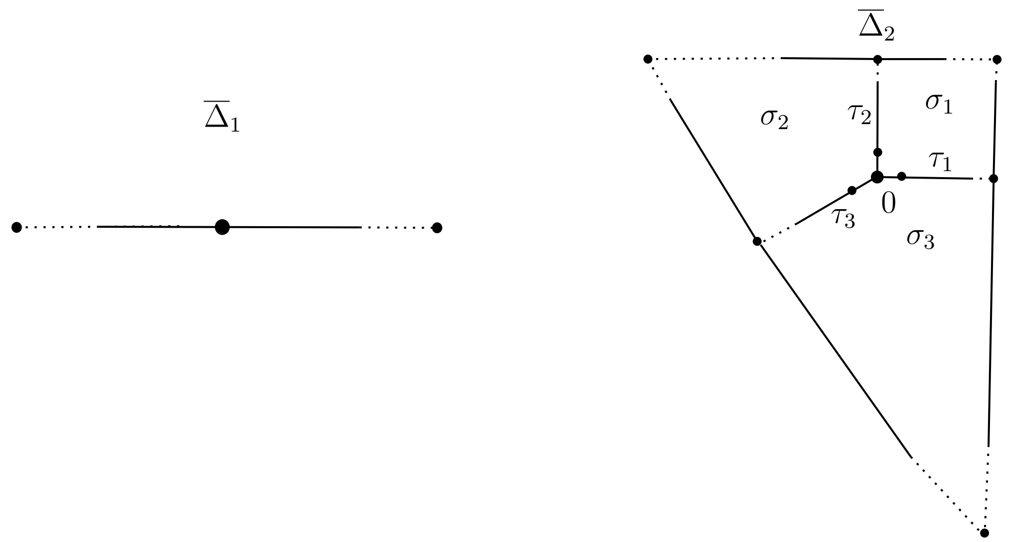


FIGURE 5. $\overline{\Delta}_1$ and $\overline{\Delta}_2$

4. Monoid Schemes

The theory of schemes was traditionally adapted for commutative rings, as described by [DF04]. Here however, we want to translate those ideas for monoids as described in [CnHWW15].

4.1. **Pointed Toric Monoids.** We will work throughout with pointed toric monoids. A *pointed monoid* is a monoid M containing a unique totally absorbing element, denoted ∞. A *totally absorbing element* of a monoid has the property that for all $m \in M$, $\infty + m = m + \infty = \infty$. From any monoid M we can construct a pointed monoid $M^\infty = M \cup \infty$ by including an absorbing element. A *pointed toric monoid* is a pointed monoid $M = P^\infty$ produced by including an absorbing element to a toric monoid P or, equivalently, a monoid for which the underlying unpointed monoid $P = M - \{\infty\}$ is toric. A *morphism of pointed monoids* is a homomorphism of monoids sending 0 to 0 and ∞ to ∞.

Recall that in our definition of ideal in Section 2.1, the emptyset is always vacuously a prime ideal of a monoid. However, we have the following lemma:

Lemma 4.1. *Let M be a pointed toric monoid and $I \subseteq M$ a non-empty ideal. Then $\infty \in I$.*

<u>Proof:</u> Let I be an ideal and $i \in I$. Then $i + \infty = \infty \in I$.$\square$

In particular, $\{\infty\}$ is a prime ideal of any pointed monoid, and in the geometry of pointed monoids usually plays the role of the minimal prime ideal. With this in mind, we follow the conventions of [CnHWW15] and define ideals of pointed toric monoids slightly differently: let M be a pointed toric monoid; an *ideal* $N \subset M$ is a proper pointed subset, meaning $M \neq N$ and $\infty \in N$, such that N is closed under the operation of M and for all $n \in N$ and $m \in M$, $m \cdot n \in N$. Thus ideals of pointed toric monoids are by definition nonempty.

We use the notation $\mathrm{Spec}(M) = \{p \subset M | p \text{ is a prime ideal}\}$ for the prime spectrum of a pointed monoid M. We make this set a topological space by endowing $\mathrm{Spec}(M)$ with the *Zariski topology* that defines closed sets as $Z(I) = \{p \in \mathrm{Spec}(M) \mid I \subseteq p\}$ for I an ideal in M. Then for $f \in M$, call D_f the set of prime ideals in X that do not contain f. The set D_f is called a *principal (or distinguished) open set* in X, and the distinguished opens form a basis for the Zariski topology on $\mathrm{Spec}(M)$.

In [CnHWW15], localization of a monoid at a prime is written as M_p for p a prime ideal. In order to avoid confusion, we will always write the localization at a prime as $M[\frac{1}{M-p}]$.

Remark 1. *For a monoid M, it is generally the case that $(M^\infty)^k \neq (M^k)^\infty$ for $k > 1$.*

The former has an absorbing element in each of the $k-$dimensions and the latter has a single absorbing element. For example, in $\mathrm{Spec}((\mathbb{N}^3)^\infty)$, any element of this pointed monoid is either a 3-tuple whos coordinates are elements of $\mathbb{N}$ or the absorbing element ∞. In $(\mathbb{N}^\infty)^3$, any element of this monoid are 3-tuples whos coordinate are elements of $\mathbb{N}^\infty$. The absorbing elements of $(\mathbb{N}^\infty)^k$ are $(\infty, 0, 0), (0, \infty, 0)$ and $(0, 0, \infty)$.

4.2. **Sheaves.** Let X be a topological space. A *presheaf* $\mathcal{F}$ on X is comprised of a monoid of functions $\mathcal{F}(U)$ on U for every open set $U \subset X$ and the property that for inclusions of open sets $U \subseteq V$, there exists a monoid homomorphism given by the restriction map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ satisfying:

(1) $r_{U,U} = \mathrm{Id}|_U$
(2) for open subsets $U \subset V \subset W$, $r_{W,U} = r_{W,V} \circ r_{V,U}$.

An element $f \in \mathcal{F}(U)$ is called a *section*. An element $g \in \mathcal{F}(X)$ is called a *global section*. A presheaf $\mathcal{F}$ on X is a *sheaf* if the following axioms are satisfied: For $\bigcup_i U_i$ an open covering of U:

(3) Locality: if $s, t \in \mathcal{F}(U)$ and $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.
(4) Gluing: if $s_i \in \mathcal{F}(U_i)$ and $U_i \cap U_j = \emptyset$ such that $r_{U_i, U_i \cap U_j}(s_i) = r_{U_j, U_i \cap U_j}(s_j)$ for all i, j, then there exists $s \in \mathcal{F}(U)$ such that $r_{U,U_i}(s) = s_i$ for all i.

As a quick example, let U in Figure 6 be covered by U_1, U_2 and U_3 (colored red, blue and green respectively). Suppose we have functions f_1, f_2, f_3 for corresponding U_i, as well as for each $U_i \cap U_j$, $f_i = f_j$. The gluing axiom guarantees there exists a function $f \in \mathcal{O}(U)$ such that $r_{U,U_i}(f) = f_i$.

Let $\mathcal{F}$, $\mathcal{G}$ be sheaves of monoids of a topological space X. A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is a family of monoid homomorphisms, one for each open set of X that respects restrictions. For an open set $U \subseteq X$, we get a homomorphism

$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$. Consider another open set $U' \subseteq U \subseteq X$. Then we expect $r|_{U,U'} \circ \varphi(U) = r|_{U,U'} \circ \varphi(U')$, resulting in a commutative diagram.

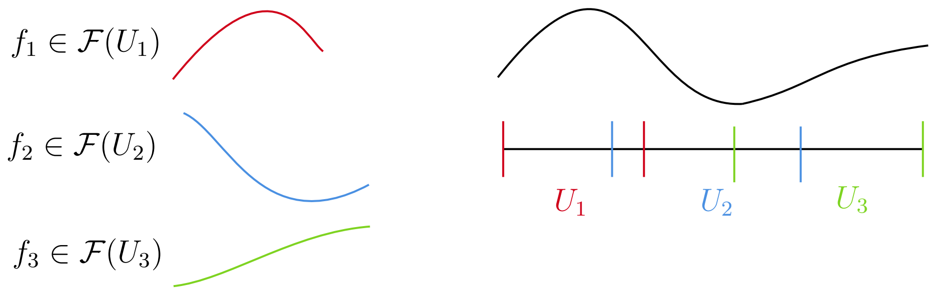


FIGURE 6. The gluing of functions f_1, f_2 and f_3

4.3. **Schemes.** An important example is the structure sheaf we build for any space $X = \mathrm{Spec}(M)$ constructed as the prime spectrum of a pointed monoid. The structure sheaf of X is defined on distinguished open sets to be $\mathcal{O}_X(D_f) = M[\frac{1}{f}]$. Global sections of the structure sheaf are $\mathcal{O}_X(\mathrm{Spec}(M)) = M$. Since distinguished open sets form a basis, the sheaf axioms uniquely determine this structure sheaf on every open set of X. Briefly, on an arbitrary open set $U \subset X$, the local sections in $O_X(U)$ take the form of the subset of $\Pi_{p\in U} M[\frac{1}{M-p}]$ consisting of elements which locally look like $\frac{a}{f^k}$ for some $f \in M$. For $p \in \mathrm{Spec}(M)$ we call the localized monoid $M[\frac{1}{M-p}]$ the *stalk* (or *local monoid*) of the structure sheaf at p. For more details, see [CnHWW15, Section 1] (or [DF04, Section 15.5] for the theory over rings). We call such a space $X = \mathrm{Spec}(M)$ with its structure sheaf an *affine monoid scheme.*

Familiarizing ourselves with the affine monoid scheme $X = \mathrm{Spec}((\mathbb{N}^k)^\infty)$ is essential for the rest of the paper. Denote $\mathbb{N}(n_1, n_2, ..., n_k)$ as the string of products of $\mathbb{N}$ and $\mathbb{N}^+$ where $n_i = 0$ (or 1) represents $\mathbb{N}$ (or $\mathbb{N}^+$) in the $i-$th coordinate, and let $\mathbb{N}(n_1, n_2, ..., n_k)^\infty = \mathbb{N}(n_1, n_2, ..., n_k) \cup \{\infty\}$. For example, $\mathbb{N}(1, 0, 1) = \mathbb{N}^+ \times \mathbb{N} \times \mathbb{N}^+$ and $\mathbb{N}(1, 1, 0)^\infty = \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N} \cup \{\infty\}$. Also, allow e_i to denote the primitive $i - th$ coordinate vector of $\mathbb{N}^k$.

The prime ideals in $\mathbb{N}^\infty$ are simply $\{\infty\}$ and $\mathbb{N}^+ \cup \infty$. In $(\mathbb{N}^2)^\infty$, the prime ideals are $\{\{\infty\}, \mathbb{N}(1, 0)^\infty, \mathbb{N}(0, 1)^\infty, \mathbb{N}^2 - (0, 0) \cup \{\infty\}\}$, as pictured in 4.3. Notice that the arrows are squiggly. This signifies that the topology has partial ordering: for $x, y \in X$, $x \leq y$ means y is in the closure of $\{x\}$.

Lemma 4.2. *The sets* $\mathbb{N}(n_1, n_2, ..., n_k)$ *are prime ideals in* $\mathbb{N}^k$ *when at least one* $n_i = 1$ *and at least one* $n_j = 0$.

Proof: Suppose $\mathbb{N}(n_1, n_2, ..., n_k)$ has a 1 in the $j-$th positions. Assume $(a_1, a_2, ..., a_k)+ (b_1, b_2, ..., b_k) \in \mathbb{N}(n_1, n_2, ..., n_k)$ and $(a_1, a_2, ..., a_k) \notin \mathbb{N}(n_1, n_2, ..., n_k)$. If $(a_1, a_2, ..., a_k) \notin \mathbb{N}(n_1, n_2, ..., n_k)$, it must mean that $a_j = 0$. But for $a_j + b_j \in \mathbb{N}(\text{of } j - \text{th dimension})$

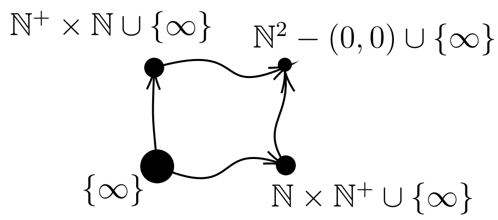


FIGURE 7. Prime spectrum of $(\mathbb{N}^2)^\infty$

which is just $\mathbb{N}^+$, then $b_j = 1$. So $(b_1, b_2, ..., b_k) \in \mathbb{N}(n_1, n_2, ..., n_k)$.□

Lemma 4.3. $\mathbb{N}^k - (0, 0, ..., 0)$ *is a prime ideal in* $\mathbb{N}^k$.

Proof: Let $a + b \in \mathbb{N}^k - (0, 0, ..., 0)$ and assume $a \notin \mathbb{N}^k - (0, 0, ..., 0)$. Then $a = (0, 0, ..., 0)$ which is the identity of $\mathbb{N}^k$. So $a + b = 0 + b = b \in \mathbb{N}^k - (0, 0, ..., 0)$.□

It is straightforward to see that the analogous lemma's hold for $\mathbb{N}(n_1, n_2, ..., n_k)^\infty$ and $\mathbb{N}^k - (0, 0, ..., 0) \cup \{\infty\}$ in $(\mathbb{N}^k)^\infty$.

We call the prime ideal in Lemma 4.2 the *closed* or *maximal* point of $\mathrm{Spec}(\mathbb{N}^k)$ since it contains every other ideal. Likewise, we call $\emptyset$ the *minimal* point of $\mathrm{Spec}(\mathbb{N}^k)$ it is contained is every other ideal.

Lemma 4.4. *Let X be a monoid scheme and $U \subseteq X$ an open subscheme. Then the following are equivalent.*
(1) U is an affine monoid scheme
(2) U has a unique maximal point
If $X = Spec(A)$, every affine open subscheme is $Spec(A_p)$ where $A_p = A\left[\frac{1}{A-p}\right]$ for some p.

A proof of this lemma can be found in [CnHWW15]. We now put Lemma 4.4 into practice using $X = \mathrm{Spec}((\mathbb{N}^2)^\infty)$. Let $U = \{\{\infty\}, \mathbb{N}(1,0)^\infty\}$. This is open since this is the distinguished open set of the ideal $\langle e_2 \rangle$. By Lemma 4.4, there must exist a prime ideal p such that $U = \mathrm{Spec}(\mathbb{N}^2[\frac{1}{\mathbb{N}^2 - p}]^\infty)$. One way to obtain this affine monoid scheme is abuse the fact that U was defined from the distinguished open set of the ideal $\langle e_2 \rangle$. Thus let's localize e_2 in our monoid. That is, $\mathbb{N}^2[e_2^{-1}]^\infty = \mathbb{N}^{2\infty}_{e_2} = \mathbb{N} \times \mathbb{Z}^\infty$. The prime ideals in $\mathbb{N} \times \mathbb{Z}^\infty$ are $\{\infty\}$ and $\mathbb{N}^+ \times \mathbb{Z}^\infty$ as pictured below. Notice that $\{\infty\} \subseteq \{\infty\}$ and $\mathbb{N} \times (\mathbb{N}^+)^\infty \subseteq \mathbb{Z} \times (\mathbb{N}^+)^\infty$

4.4. **Morphisms.** Let $(\mathrm{Spec}(M), \mathcal{O}_{\mathrm{Spec}(M)})$ and $(\mathrm{Spec}(N), \mathcal{O}_{\mathrm{Spec}(N)})$ be two affine monoid schemes. A *morphism of affine schemes*

$$\varphi : (\mathrm{Spec}(M), \mathcal{O}_{\mathrm{Spec}(M)}) \to (\mathrm{Spec}(N), \mathcal{O}_{\mathrm{Spec}(N)})$$

is a pair $\varphi = (\varphi, \varphi^\#)$ such that

(1) $\varphi : \mathrm{Spec}(M) \to \mathrm{Spec}(N)$ is Zariski continuous map of topological spaces;
(2) there are monoid homomorphisms $\varphi^\#(U) : \mathcal{O}_{\mathrm{Spec}(N)}(U) \to \mathcal{O}_{\mathrm{Spec}(M)}((\varphi)^{-1}(U))$ for every Zariski open subset U in $\mathrm{Spec}(M)$ that commute with restrictions maps, as in Section 4.2, and;

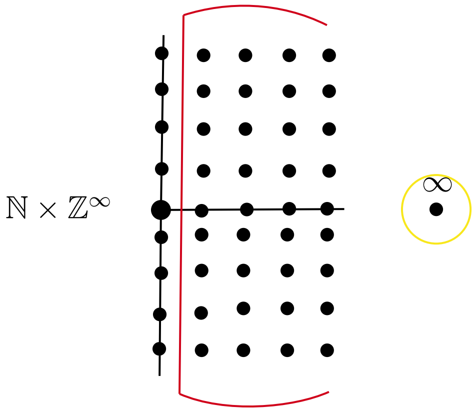


FIGURE 8. $\mathbb{N} \times \mathbb{Z}^\infty$ and it's prime ideals

(3) the induced maps $M[\frac{1}{M-p}] \longrightarrow N[\frac{1}{N-\varphi(p)}]$ for each $p \in \mathrm{Spec}(M)$ send the maximal ideal of $M[\frac{1}{M-p}]$ into the maximal ideal of $N[\frac{1}{N-\varphi(p)}]$.

Though we don't get into the technical details here, briefly, the system of morphisms in (2) makes $\varphi^\#$ a morphism of sheaves $\mathcal{O}_{\mathrm{Spec}(N)} \longrightarrow \varphi_* \mathcal{O}_{\mathrm{Spec}(M)}$ on $\mathrm{Spec}(N)$ as in Section 4.2 above. Notice how if we choose $U = \mathrm{Spec}(N)$, then $\varphi^{-1}(U) = \mathrm{Spec}(M)$. This gives us a monoid homomorphism

$$\varphi^\# : \mathcal{O}_{\mathrm{Spec}(M)}(\mathrm{Spec}(M)) \to \mathcal{O}_{\mathrm{Spec}(N)}(\mathrm{Spec}(N))$$

on global sections, i.e. a monoid homomorphism $\varphi^\# : N \to M$. Just as in scheme theory for rings, this definition makes the category of affine monoid schemes equivalent to the category of pointed monoids with morphisms going in the opposite direction. In the other direction, from a morphism $\varphi^\# : N \to M$ we can build the corresponding Zariski continuous map $\varphi : \mathrm{Spec}(M) \to \mathrm{Spec}(N)$ of topological spaces by sending the prime ideal $p \subset M$ to its inverse image $(\varphi^\#)^{-1}(p)$, a prime ideal in N. Throughout the rest of this thesis we will use this equivalence to study morphisms of affine schemes by studying the underlying morphisms of pointed monoids.

5. Morphisms of Affine Spaces

We now begin our study of sets of morphisms in the categories of pointed toric monoids, toric monoid schemes, and extended fans.

5.1. Morphisms from $\mathbb{A}^1$ to $\mathbb{A}^1$.

We begin with morphisms from the affine line to the affine line in each of the three settings.

1. $\mathrm{Hom}_{\mathrm{Mon}}(\mathbb{N}^\infty, \mathbb{N}^\infty)$: A morphism in this hom set is determined by what the morphism does to the element 1 in $\mathbb{N}^\infty$, the domain, since 0 must be sent to 0 and ∞ must go to ∞. Then 1 can go to either any element of $\mathbb{N}$ or ∞. Thus $\mathrm{Hom}_{Mon}(\mathbb{N}^\infty, \mathbb{N}^\infty) = \mathbb{N} \cup \{\mathrm{pt}\} = \mathbb{N}^\infty$ as sets.

2. $\mathrm{Hom}_{\mathrm{ERPF}}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{\mathbb{N}^\infty})$: By the structure lemma for morphisms of extended cones, a morphism $\overline{\sigma}_1 \longrightarrow \overline{\sigma}_2$ can be uniquely factored into a morphism from $\overline{\sigma}_1$ to $\overline{F}(\sigma_2, \tau)$ for some τ a face of σ_2 whose image meets the open interior of

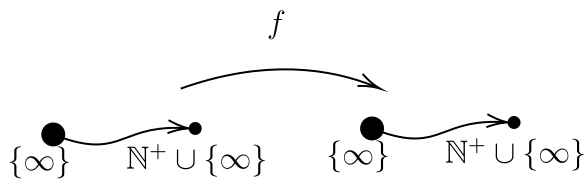


FIGURE 9. $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^{\infty}), \mathrm{Spec}(\mathbb{N}^{\infty}))$

$\overline{F}(\sigma_2, \tau)$, followed by an inclusion into $\overline{\sigma}_2$. We have two faces of σ_2 to choose from: $\tau = 0$ and $\tau = \sigma_{\mathbb{N}^\infty}$.

1) For $\tau = 0$ the face $F(\overline{\sigma}_{\mathbb{N}^\infty}, 0)$ is simply $\mathrm{Hom}(\mathbb{N}, \mathbb{R}_{\geq 0})$. Now we can ask what are the morphisms from $\sigma_{\mathbb{N}}$ to $\mathrm{Hom}(\mathbb{N}, \mathbb{R}_{\geq 0})$. The morphism must respect the underlying lattices of these cones, as well as preserve orientation in terms of positive and negative. The lattices for each of these cones is $\mathbb{Z}$, so now we can consider the maps $\mathbb{Z} \to \mathbb{Z}$. As before, it matters only where 1 gets sent to but now we would like to send it somewhere positive. However sending 1 to 0 would collapse $\mathbb{Z}$ onto 0, which is also an option. Thus we get $\mathbb{N}$ choices for these morphisms.
2) The face $F(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{\mathbb{N}^\infty})$ is simply a point. There is only one map from $\overline{\sigma}_{\mathbb{N}^\infty}$ to a point, the constant map.

We get that the Hom set of $\overline{\sigma}_{\mathbb{N}^\infty}$ to itself is also $\mathbb{N}^{\infty}$.

3. $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^{\infty}), \mathrm{Spec}(\mathbb{N}^{\infty}))$: The discrete geometry in this case is diagramed in Figure 9. The set $\mathrm{Spec}(\mathbb{N}^{\infty})$ consists of the points $\{\infty\}$ and $(\mathbb{N}^{+})^{\infty}$. The closure of $\{\infty\}$ is the whole space, since $\{\infty\}$ is the minimal point. There are 3 continuous maps of topological spaces: (1) the map that sends $\mathrm{Spec}(\mathbb{N}^{\infty})$ to $\{\infty\}$, (2) $\{\infty\}$ to $\{\infty\}$ and $(\mathbb{N}^{+})^{\infty}$ to $(\mathbb{N}^{+})^{\infty}$ and (3) $\mathrm{Spec}(\mathbb{N}^{\infty})$ to $(\mathbb{N}^{+})^{\infty}$. We split this into several subcases, wherein we will consider the possible morphisms of underlying structure sheaves, $f^{\#}$, for each f. Unsurprisingly, $f^{\#}$ is defined from $\mathbb{N}^{\infty} \to \mathbb{N}^{\infty}$.

<u>Subcase 1</u> Since $f(\{\infty\}) = \{\infty\}$ and $f(\mathbb{N}^{\infty}) = \{\infty\}$, we see that $f^{\#-1}(\{\infty\}) = \{\infty\}$ and $f^{\#-1}(\mathbb{N}^{\infty}) = \{\infty\}$. The induced morphism $f^{\#}$ must send $0 \mapsto 0$ and $\infty \mapsto \infty$. Thus what determines our map is where e_1 is sent. To preserve the fact that no positive integers are sent to ∞, it's clear that $e_1 \mapsto 0$. Thus we have a constant zero map (away from ∞).

<u>Subcase 2</u> Since $f(\{\infty\}) = \{\infty\}$ and $f((\mathbb{N}^{+})^{\infty}) = (\mathbb{N}^{+})^{\infty}$, then $f^{\#-1}(\{\infty\}) = \{\infty\}$ and $f^{\#-1}((\mathbb{N}^{+})^{\infty}) = (\mathbb{N}^{+})^{\infty}$. To guarantee that $(\mathbb{N}^{+})^{\infty}$ is mapped onto $(\mathbb{N}^{+})^{\infty}$, then $e_1 \mapsto a \in \mathbb{N}^{+}$. This morphism f induces countably many morphisms $f^{\#}$, one for each choice of a.

<u>Subcase 3</u> If $f(\{\infty\}) = (\mathbb{N}^{+})^{\infty}$ and $f((\mathbb{N}^{+})^{\infty}) = (\mathbb{N}^{+})^{\infty}$, then $f^{\#-1}(\{\infty\}) = (\mathbb{N}^{+})^{\infty}$ and $f^{\#-1}((\mathbb{N}^{+})^{\infty}) = (\mathbb{N}^{+})^{\infty}$. For this map, $f^{\#}$ must send $e_1 \mapsto \infty$. This is the constant ∞ map (away from zero).

Since we have a constant zero map, countably many maps for each positive integer and a constant ∞ map, this Hom set is $\mathbb{N}^\infty$. Thus we get the Hom sets of $\mathrm{Hom}_{Mon}(\mathbb{N}^\infty, \mathbb{N}^\infty)$, $\mathrm{Hom}_{ERPF}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{\mathbb{N}^\infty})$ and $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathrm{Spec}(\mathbb{N}^\infty))$ are all the same.

5.2. **Morphisms from $\mathbb{A}^1$ to $\mathbb{A}^2$.** We now compute one more example of this equivalence of Hom sets in the affine setting. We'll show that Hom sets of $\mathrm{Hom}_{Mon}((\mathbb{N}^2)^\infty, \mathbb{N}^\infty)$, $\mathrm{Hom}_{ERPC}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{(\mathbb{N}^2)^\infty})$ and $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty)), \mathrm{Spec}((\mathbb{N}^2)^\infty))$ again have a similar structure.

1. $\mathrm{Hom}_{Mon}((\mathbb{N}^2)^\infty, \mathbb{N}^\infty)$: Unstrikingly, this is the simplest Hom set to dissect. However, it will give us an early answer to seek out when we try to describe Hom sets for $\mathrm{Hom}_{ERPC}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{(\mathbb{N}^2)^\infty})$ and $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathrm{Spec}((\mathbb{N}^2)^\infty))$. As with any morphism of pointed monoids, $0 \mapsto 0$ and $\infty \mapsto \infty$. So what is left is to determine where e_1 and e_2 go. There are two $\mathbb{N}^\infty$ worth of choices, thus $\mathrm{Hom}_{Mon}(\mathrm{Spec}((\mathbb{N}^2)^\infty), \mathbb{N}^\infty) = (\mathbb{N}^\infty)^2$. Note that, as in Remark 4.1 above, $(\mathbb{N}^\infty)^2 \neq (\mathbb{N}^2)^\infty$.

2. $\mathrm{Hom}_{ERPC}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{(\mathbb{N}^2)^\infty})$: We again use the unique factorization of morphisms of extended cones to classify the possible maps. There are four faces of $\sigma_{\mathbb{N}^\infty}$ (as depicted in Figure 2), so four possible faces whose extension we may factor through: $F(\sigma_{\mathbb{N}^2}, \tau_1)$, $F(\sigma_{\mathbb{N}^2}, \tau_2)$, $F(\sigma_{\mathbb{N}^2}, 0)$ and $F(\sigma_{\mathbb{N}^2}, \sigma_{\mathbb{N}^2})$.

   <u>Subcase 1</u> In the $F(\sigma_{\mathbb{N}^2}, \tau_1)$ case, this is simply the different ways we can map $\sigma_{\mathbb{N}}$ to $\sigma_{\mathbb{N}}$. We think of $F(\sigma_{\mathbb{N}^2}, \tau_1)$ as the vertical line at (∞, b) for $b \in \mathbb{N}$. Recall we'd like to preserve the underlying lattice of the vector space. So 1 must go to some element of $\mathbb{N}$. Thus there are $\mathbb{N}$ choices.

   <u>Subcase 2</u> The $F(\sigma_{\mathbb{N}^2}, \tau_2)$ is nearly identical to the previous case. However, $F(\sigma_{\mathbb{N}^2}, \tau_2)$ represents the horizontal line at (a, ∞) for $a \in \mathbb{N}$ of the extended cone $\overline{\sigma}_{(\mathbb{N}^2)^\infty}$. We get $\mathbb{N}$ maps.

   <u>Subcase 3</u> We now consider the maps from $\sigma_{\mathbb{N}}$ to $F(\sigma_{\mathbb{N}^2}, 0)$ which is just $\sigma_{\mathbb{N}^2}$ itself. Again, to preserve the underlying lattice and vector space structure, 1 must go to a lattice point of $\sigma_{\mathbb{N}^2}$. So we get $\mathbb{N}^2$ possible morphisms, one for each lattice point.

   <u>Subcase 4</u> The face $F(\sigma, \sigma)$ is a single point. So this gives just a constant map.

   Thus we get the followings maps: $\mathbb{N}^2, (\infty, \mathbb{N}), (\mathbb{N}, \infty)$ and (∞, ∞). Thus we get that $\mathrm{Hom}_{ERPC}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{(\mathbb{N}^2)^\infty}) = (\mathbb{N}^\infty)^2$.

3. $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathrm{Spec}((\mathbb{N}^2)^\infty))$: As before, we will first consider all the continuous maps of topological spaces, and then classify the corresponding underlying morphism of sheaves.

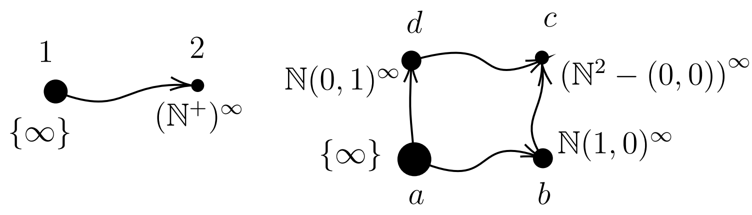


FIGURE 10. We will refer to this diagram to describe the continuous maps. The convention used to describe maps will number preimage points and letter image points. A pair, consisting of a number and a letter, is what f if the letter to right of the number. For example, if f has the representation $1d3e$, then f sends the point representing 1 (or 3) to the point representing d (or e).

The continuous maps of spaces are: 1a2a, 1a2b, 1a2d, 1a2c, 1b2c, 1b2b, 1c2c, 1d2c and 1d2d (using notation as described in Figure 10). As always, the underlying morphism of sheaves will determine the classification of associated morphisms of schemes for each continuous map of topological spaces. To determine said morphisms from $(\mathbb{N}^2)^\infty$ to $\mathbb{N}^\infty$, we only need to consider where e_1 and e_2 go. Thus for each subcase, we will identify the corresponding maps by (x, y) where e_1 is sent to x and e_2 is sent to y. When we say preimage, we mean the preimage of the induced morphism $f^\#$ of f.

Subcase 1: 1a2a Since the preimage of $\{\infty\}$ and $(\mathbb{N}^+)^\infty$ is just ∞, e_1 and e_2 must go to 0. Thus we will identify this constant map by (0,0).

Subcase 2: 1a2b The preimage of $\{\infty\}$ and $(\mathbb{N}^+)^\infty$ is $\{\infty\}$ and $\mathbb{N}(1,0)^\infty$ respectively. This means $e_2 \mapsto 0$ and e_1 is sent to an element of $\mathbb{N}^+$. Thus 1a2b corresponds to maps $(a, 0)$ for $a \in \mathbb{N}^+$.

Subcase 3: 1a2d By similar analogy, $e_1 \mapsto 0$ and e_2 is sent to an element of $\mathbb{N}^+$. Thus 1a2d corresponds to maps $(0, b)$ for $b \in \mathbb{N}^+$.

Subcase 4: 1a2c The preimage of $\{\infty\}$ is $\{\infty\}$ while the preimage of $(\mathbb{N}^+)^\infty$ is $(\mathbb{N}^2 - 0)^\infty$. So e_1 and e_2 are sent to nonzero elements of $\mathbb{N}^+$. So 1a2c corresponds to maps $(a, b) \in \mathbb{N}(1, 1)$.

Subcase 5: 1b2c Since the preimage of $\{\infty\}$ is $\mathbb{N}(1, 0)^\infty$ and preimage of $(\mathbb{N}^+)^\infty$ is $(\mathbb{N}^2 - 0)^\infty$, $e_1 \mapsto \infty$ and $e_2 \mapsto b > 0$. So 1b2c corresponds to maps (∞, b) for $b \in \mathbb{N}^+$.

Subcase 6: 1b2b The preimage of $\{\infty\}$ is $\mathbb{N}(1, 0)^\infty$ and $(\mathbb{N}^+)^\infty$ is $\mathbb{N}(1, 0)^\infty$. Thus this map sends $e_1 \mapsto \infty$ and $e_2 \mapsto 0$. Thus 1b2b corresponds to the morphism $(\infty, 0)$.

Subcase 7: 1c2c The preimage of $\{\infty\}$ and $(\mathbb{N}^+)^\infty$ is $(\mathbb{N}^2 - 0)^\infty$. Thus 1c2c corresponds to the morphism (∞, ∞).

<u>Subcase 8: 1d2c</u> The preimage of $\{\infty\}$ is $\mathbb{N}(0,1)^\infty$ and $(\mathbb{N}^+)^\infty$ is $(\mathbb{N}^2-0)^\infty$. This corresponds to maps (a,∞) for $a \in \mathbb{N}^+$.

<u>Subcase 9: 1d2d</u> Finally, the preimate of $\{\infty\}$ is $\mathbb{N}(0,1)^\infty$ and preimage of $(\mathbb{N}^+)^\infty$ is $\mathbb{N}(0,1)^\infty$. This corresponds to the map $(0,\infty)$.

Rearranging the maps in the following combinatorial diagram reveals that the Hom set from $\mathrm{Spec}(\mathbb{N}^\infty)$ to $\mathrm{Spec}((\mathbb{N}^2)^\infty)$ is simply $(\mathbb{N}^\infty)^2$.

$$\begin{array}{ccc}
(0,\infty)^\bullet & (a,\infty)^\bullet & (\infty,\infty)^\bullet \\
(0,b)^\bullet & (a,b)^\bullet & (\infty,b)^\bullet \\
(0,0)^\bullet & (a,0)^\bullet & (\infty,0)^\bullet
\end{array}$$

Figure 11. $(\mathbb{N}^\infty)^2$

We present the generalized computation of the Hom set of scheme morphisms from $\mathrm{Spec}(\mathbb{N}^\infty)$ to $\mathrm{Spec}((\mathbb{N}^k)^\infty)$.

Theorem 5.1. *Hom*(*Spec*$(\mathbb{N}^\infty)$, *Spec*$((\mathbb{N}^k)^\infty) = (\mathbb{N}^\infty)^k$

Proof. This proof is combinatorial and generalizes the processes used in 5.1 and 5.2. As always we will think of mapping the point labeled 1 ($\{\infty\} \in \mathrm{Spec}(\mathbb{N}^\infty)$) to some prime ideal in $\mathrm{Spec}((\mathbb{N}^k)^\infty)$ and 2 ($(\mathbb{N}^+)^\infty \in \mathrm{Spec}(\mathbb{N}^\infty)$) to some prime ideal in $\mathrm{Spec}((\mathbb{N}^k)^\infty)$ that respects partially ordering, as mentioned earlier.

First, we map 1 to $\{\infty\} \in \mathrm{Spec}((\mathbb{N}^k)^\infty)$. Then 2 is allowed to go to any other prime ideal in $\mathrm{Spec}((\mathbb{N}^k)^\infty)$, since any other prime ideal in $\mathrm{Spec}((\mathbb{N}^k)^\infty)$ contains ∞, by Proposition 4.3. The way we will denote this is $2 \mapsto \mathbb{N}(n_1, n_2, ..., n_k)^\infty$ where $n_i = 0$ for at least one i. As always, this is a single morphism of topological spaces, but it may be induced by many underlying morphisms of pointed monoids. If 2 is sent to $\{\infty\}$, then this results in the zero morphism $(0,0,...,0)$. Otherwise, we see that $f^{\#-1}(\{\infty\}) = \{\infty\}$ and $f^{\#-1}((\mathbb{N}^+)^\infty) = \mathbb{N}(n_1, n_2, ..., n_k)^\infty$. Next we consider where $e_1, ..., e_k$ get sent. If $n_1 = 1$, then $e_1 \mapsto a_1$. Generally, if $n_i = 1$, then $e_i \mapsto a_i \in \mathbb{N}^+$, otherwise $e_i \mapsto 0$. We can explicitly write this as

$$f : 1 \mapsto \infty, 2 \mapsto \mathbb{N}(n_1, n_2, ..., n_k)^\infty \Leftrightarrow f^\# \text{ defined by } (a_1\delta_{1,n_1}, a_2\delta_{1,n_2}, ..., a_k\delta_{1,n_k})$$

where $\delta_{a,b}$ is the Kronecker delta. Thus the maps we get are all positive integral linear combinations of $(1,0,...,0), (0,1,0,...,0), ...,$ and $(0,...,0,1)$ where at least one direction is excluded. For example, we obtain the maps $(a_1,0,...,0)$, $(a_1,a_2,0,...,0)$, $(0,a_2,0...,0,a_6,...,a_k)$ and $(a_1,0,a_3,a_4,...,a_{k-1},a_k)$. We exclude the case where all $n_i = 1$ because that is not a prime ideal. Instead, we have the maximal ideal $(\mathbb{N}^k - 0)^\infty$. If we send $(\mathbb{N}^+)^\infty$ to this prime ideal, we may send each e_i to an element of $\mathbb{N}^+$. This provides the maps $(a_1, a_2, ..., a_k)$.

Next we consider point 1 being sent to $\mathbb{N}(n_1, n_2, ..., n_k)^\infty$. We must classify all the possible targets for the point 2.

To generalize those details, let $1 \mapsto \mathbb{N}(n_1, n_2, ..., n_k)^\infty$, where at least one entry is 1 and one is 0, and $2 \mapsto \mathbb{N}(m_1, m_2, ..., m_k)^\infty$ where again at least one entry is 1 and one is 0, and moreover if $n_i = 1$ then $m_i = 1$ for all i. Then the possible maps $f^\#$ are defined by

$$\left(\infty\delta_{1,\delta_{n_1,m_1}} + a_1\delta_{0,\delta_{n_1,m_1}}, ..., \infty\delta_{1,\delta_{n_k,m_k})} + a_k\delta_{0,\delta_{n_k,m_k}}\right)$$

where $a_i \in \mathbb{N}^+$. So some maps we get are $(\infty, 0, ..., 0)$, $(a_1, \infty, \infty, 0, ..., a_k)$, and $(\infty, \infty, 0, ..., \infty, \infty)$. Thus we get positive integral linear combinations of $(1, 0, ..., 0)$, ..., $(0, 0, ..., 1)$ and $(\infty, 0, ..., 0), ..., (0, ..., 0, \infty)$ except where the components arn't strictly just all positive numbers, all zeros or all infinities. Moreover, we get all the possible maps that have at least one ∞ except for the constant ∞ map.

Lastly, what remains is the constant ∞ map, which is obtained by sending $1, 2 \mapsto (\mathbb{N}^k - 0)^\infty$. Thus we get the map $(\infty, ..., \infty)$ and we have all possible elements of $a \in (\mathbb{N}^\infty)^k$. □

There is an analogous proof for showing $\mathrm{Hom}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{(\mathbb{N}^k)^\infty})$ is also $(\mathbb{N}^\infty)^k$, which we refrain from writing for the sake of berevity. At this point, our understanding of $\mathrm{Hom}((\mathbb{N}^k)^\infty, \mathbb{N}^\infty)$ should be deep enough to see that this hom set is $(\mathbb{N}^\infty)^k$, since each $e_i, 1 \leq i \leq k$, has $\mathbb{N}^\infty$ choices to get send to. We end this section with the following theorem.

Theorem 5.2. *The following Hom sets are equal*

1. *Hom*$(\textit{Spec}((\mathbb{N}^k)^\infty), \mathbb{N}^\infty)$
2. *Hom*$(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\sigma}_{\textit{Spec}((\mathbb{N}^k)^\infty)})$
3. *Hom*$(\textit{Spec}(\mathbb{N}^\infty), \textit{Spec}((\mathbb{N}^k)^\infty))$

6. Morphisms of $\mathbb{A}^1_{\mathbb{F}1}$ to $\mathbb{P}^1_{\mathbb{F}1}$

We now begin translating these ideas into Hom sets involving $\mathbb{P}^1_{\mathbb{F}_1}$, projective space over $\mathbb{F}_1$. We define $\mathbb{P}^1_{\mathbb{F}_1}$ to be the gluing of two copies of Spec$(\mathbb{N}^\infty)$. Lets familiarize ourselves with Spec$(\mathbb{N}^\infty)$ a bit more.

The open sets of Spec$(\mathbb{N}^\infty)$ are $\emptyset$, itself and $\{\{\infty\}\}$. This makes sense since we get exactly the amount of distinguished open sets according to the size of Spec$(\mathbb{N}^\infty)$, by Lemma 1.3 of [CnHWW15]. Since Spec$(\mathbb{N}^\infty)$ has the Zariski topology, distinguished open sets can be identified by a corresponding element of $\mathbb{N}^\infty$. Suppose we choose 1, the generator of $\mathbb{N}^\infty$. Now we localize $\mathbb{N}^\infty$ at 1 and look at it's prime spectrum. Localizling at 1 gives

$$D_1 = \mathrm{Spec}\left(\mathbb{N}^\infty\left[\frac{1}{e_1}\right]\right) = \mathrm{Spec}(\mathbb{N}^\infty[-1]) = \mathrm{Spec}(\mathbb{Z}^\infty)$$

Lemma 6.1. Spec$(\mathbb{Z}^\infty) = \{\{\infty\}\}$.

Proof. Let p be a prime ideal in $\mathbb{Z}^\infty$. By Lemma 4.1, $\infty \in p$. Assume for contradiction that $x \in p$ and $x \neq \infty$. If $x = 0$, then $x + a \in p$ for all $a \in \mathbb{Z}$. So $p = \mathbb{Z}^\infty$. If $x \neq 0$, then $x + (-x) = 0 \in p$, so again $p = \mathbb{Z}^\infty$. $\square$

Another useful lemma we would like to introduce generalizes Lemma 4.3. This will come in handy later.

Lemma 6.2 *$M - M^*$ is a prime ideal in M where M^* is the set of units in M.*

Proof. Let $x \in M - M^*$. We first show $ax \in M - M^*$ for all $a \in M$. Assume for contradiction there exists a $y \in M$ such that $y(ax) = 1$. This implies $(ya)x = 1$, but this contradicts $x \notin M^*$. So $M - M^*$ is an ideal. Let $ab \in M - M^*$. There are 4 cases: (1) a, b are units, (2) a is a non-unit, b is a unit, (3) a is unit, b is nonunit and (4) a, b no unit. Since we have already established that $M - M^*$ is ideal, it remains to just verify case (1). If $(ab) \in M - M^*$ with a, b units, then $(ab)(b^{-1}a^{-1}) = 1$ but ab is not a unit. So $M - M^*$ is a prime ideal. $\square$

Combining the proofs of Lemma 6.1 and 6.2, one can deduce that $\text{Spec}((\mathbb{Z}^k)^\infty) = \{\{\infty\}\}$. We are now able to write out the details for how to glue two copies of $\text{Spec}(\mathbb{N}^\infty)$. Let $X = \text{Spec}(\mathbb{N}^\infty)$ and $Y = \text{Spec}(\mathbb{N}^\infty)$.

Then X has points $(\mathbb{N}^+)^\infty_X$ and $\{\infty_X\}$, while Y has points $(\mathbb{N}^+)^\infty_Y$ and $\{\infty_Y\}$. The way we will glue X and Y is along an open set. Let the generator of $\mathbb{N}^\infty_X$ be x and the generator of $\mathbb{N}^\infty_Y$ be y. Then as above $U_X = D_x = \text{Spec}(\mathbb{Z}^\infty) = \{\{\infty_X\}\}$ and $U_Y = D_y = \text{Spec}(\mathbb{Z}^\infty) = \{\{\infty_Y\}\}$, where each copy of $\mathbb{Z}$ is generated by the respective generator x or y. Topologically, we glue by setting $\{\infty_X\} = \{\infty_Y\}$. In the language of atlases, X and Y are local affine charts that overlap in the open set $U_X = U_Y$. There are inclusion maps of this overlap into each chart: $i_X : U_X \longrightarrow X$ and $i_Y : U_Y \longrightarrow Y$. Topologically these inclusion maps simply include the point $\{\infty\}$ of $\mathbb{P}^1_{\mathbb{F}^1}$ as either $\{\infty_X\}$ or $\{\infty_Y\}$. Algebraically, they correspond to the morphisms of pointed monoids

$$\begin{aligned} i_X^\# : &\mathbb{N}^\infty \longrightarrow \mathbb{Z}^\infty \\ &\infty \longmapsto \infty \\ &0 \longmapsto 0 \\ &x \longmapsto 1 \end{aligned} \qquad\qquad \begin{aligned} i_Y^\# : &\mathbb{N}^\infty \longrightarrow \mathbb{Z}^\infty \\ &\infty \longmapsto \infty \\ &0 \longmapsto 0 \\ &y \longmapsto 1. \end{aligned}$$

To complete the gluing scheme theoretically, we must describe the morphism of schemes $\gamma : U_X \longrightarrow U_Y$ along which we glue. It was described topologically above. Algebraically, it corresponds to the morphisms

$$\begin{aligned} \gamma^\# : &\mathbb{Z}^\infty \longrightarrow \mathbb{Z}^\infty \\ &\infty \longmapsto \infty \\ &0 \longmapsto 0 \\ &1 \longmapsto -1. \end{aligned}$$

In $\mathbb{P}^1_{\mathbb{F}^1}$, γ acts as a transition function on the overlap of the two affine charts. Note that $\mathbb{P}^1_{\mathbb{F}_1}$ has an underlying integer lattice of rank 1, which is isomorphic to $\mathbb{Z}$. This transition function identifies the generator x with the element $-1 \in \mathbb{Z}$ and y with the element $1 \in \mathbb{Z}$, i.e. $x = -y$. This assignment will make more sense once we investigate the following Hom Set.

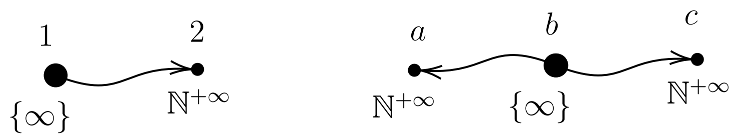


FIGURE 12. Spec($\mathbb{N}^\infty$) and $\mathbb{P}^1_{\mathbb{F}^1}$

Theorem 6.2. $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathbb{P}^1_{\mathbb{F}^1}) = (\mathbb{Z}) \cup \{\pm\infty\}$.

Proof. As above, we identify the two copies of the prime spectrum of $\mathbb{N}^\infty$ using $X = \mathrm{Spec}(\mathbb{N}^\infty)$ and $Y = \mathrm{Spec}(\mathbb{N}^\infty)$, and let U_X, U_Y be the open set consisting of $\{\infty\}$ in each corresponding affine chart. The copy of X in $\mathbb{P}^1_{\mathbb{F}^1}$ corresponds to the open set containing a and b, while Y corresponds to the open set containing b and c.

Subcase 1:1a2a This map is contained in X. In X, this map corresponded to the constant ∞ map. We call this homomorphism $-\infty$

Subcase 2: 1b2a This map is also contained in X. In X, we have already seen that this corresponds to sending $e_1 \mapsto a \in \mathbb{N}^+$. We think of these homomorphisms as the negative elements of a copy of $\mathbb{Z}$.

Notice in both these cases, the maps were completely contained in X. In other words, these maps were not glued to any map in Y. This will be useful for our construction later.

Subcase 3: 1c2c Likewise, this map is contained in Y and from Section 5.1, we know this corresponds to the constant ∞ map in y.

Subcase 4: 1b2c This corresponds to maps $b \in \mathbb{N}^+$.

Subcase 5: 1b2b This is the only case where things get interesting due to the gluing of the ∞'s of X and Y. First we will recognize the fact that Spec($\mathbb{N}^\infty$) is being sent to the subset $U_X = U_Y$ of both X and Y. From earlier, we established that the open set just containing $\{\infty\}$ corresponded to the prime spectrum of $\mathbb{Z}^\infty$. So in particular, there are two morphisms, $\alpha : \mathrm{Spec}(\mathbb{N}^\infty) \to U_X = \mathrm{Spec}(\mathbb{Z}^\infty)$ and $\beta : \mathrm{Spec}(\mathbb{N}^\infty) \to U_Y = \mathrm{Spec}(\mathbb{Z}^\infty)$. Since we glued U_X and U_Y along each other, to ultimately construct $\mathbb{P}^1_{\mathbb{F}^1}$, there is also the transition map $\gamma : U_X \to U_Y$ defined above. For the morphisms α and β to agree and determine the same map to $\mathbb{P}^1_{\mathbb{F}^1}$, they must commute with γ. This is obvious topologically (see Figure 13).

Now we discuss the underlying sheaves. Since both points of Spec($\mathbb{N}^\infty$) map to the unique point of U_X and U_Y through α and β, the preimage of N^+ in both cases is ∞_X and ∞_Y. Thus, the generators x and y of the global sections monoids $\mathbb{Z}^\infty$ of X and Y must both map to 0. From the discussion above, we saw that U_X and U_Y were glued using the map γ that sends the generator x to $-y$. This map preserves only

$$\begin{tikzcd}
 & Spec\,(\mathbb{N}^\infty) \arrow[dl, "\infty - constant\ map"'] \arrow[dr, "\infty - constant\ map"] & \\
U_X \arrow[rr, "\{\infty_x\} = \{\infty_y\}"'] & & U_Y
\end{tikzcd}$$

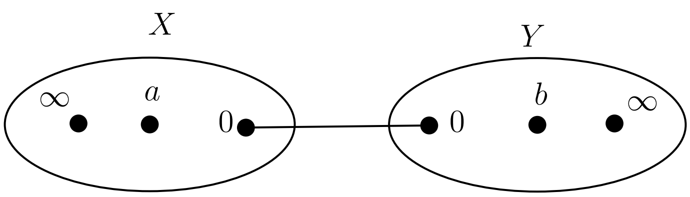
FIGURE 13. The maps α, β and γ, topologically.

$$\begin{tikzcd}
 & \mathbb{N}^\infty & \\
\mathbb{Z}^\infty \arrow[ur, "\alpha^\#"] & & \mathbb{Z}^\infty \arrow[ul, "\beta^\#"'] \arrow[ll, "\gamma^\#"]
\end{tikzcd}$$

FIGURE 14. Associated morphisms of pointed monoids $\alpha^\#, \beta^\#$ and $\gamma^\#$.

0 and ∞. So we see that diagrams are actually commutative triangles, in particular $\gamma \circ \alpha = \beta$ (as depicted in Figure 14). Thus there is one morphism of sheaves according to 1b2b, the constant 0 map. We call this map 0 and we get the Hom set given by the union of $-\infty$ and ∞ with two copies of $\mathbb{N}^+$ and a 0. This results in a copy of $\mathbb{Z} \cup \{\pm\infty\}$. $\square$

We can illustrate this Hom set as a graph, where each vertex is labeled according to where e_1 is sent in the underlying morphism of monoids and where an edge between vertices means those maps are glued together.



7. MORPHISMS OF $\mathbb{A}^1_{\mathbb{F}1}$ TO $\mathbb{P}^2_{\mathbb{F}1}$

Now, we repeat the computation for a target $\mathbb{P}^2_{\mathbb{F}1}$, the gluing of three copies of $\mathrm{Spec}((\mathbb{N}^2)^\infty)$ along open sets. Again we can study distinguished open set in $\mathrm{Spec}((\mathbb{N}^2)^\infty)$ to help us understand how these three copies are glued. Consider the distinguished open set corresponding to e_1; we invert e_1, getting the monoid $\mathbb{Z} \times \mathbb{N}^\infty$. We refer to Lemma 6.2. We cannot use Lemma 3.1 since 0 is not the only unit of $\mathbb{Z} \times \mathbb{N}^\infty$. In fact the rest of units are of the form $(a, 0)$ where $a \in \mathbb{Z}$. So the only prime ideals in $\mathbb{Z} \times \mathbb{N}^\infty$ are $\{\infty\}$ and $\mathbb{Z} \times (\mathbb{N}^+)^\infty$. Then $\mathrm{Spec}(\mathbb{Z} \times \mathbb{N}^\infty)$ corresponds to the subset $\{\{\infty\}, \mathbb{N} \times (\mathbb{N}^+)^\infty\}$ of $\mathrm{Spec}((\mathbb{N}^2)^\infty)$, since $\mathbb{N} \times (\mathbb{N}^+)^\infty \subseteq \mathbb{Z} \times (\mathbb{N}^+)^\infty$. By a similar construction, the distinguished open set corresponding to e_2 is the subset $\{\{\infty\}, \mathbb{N}^+ \times \mathbb{N}^\infty\} \subset \mathbb{N}^+ \times \mathbb{Z}^\infty$.

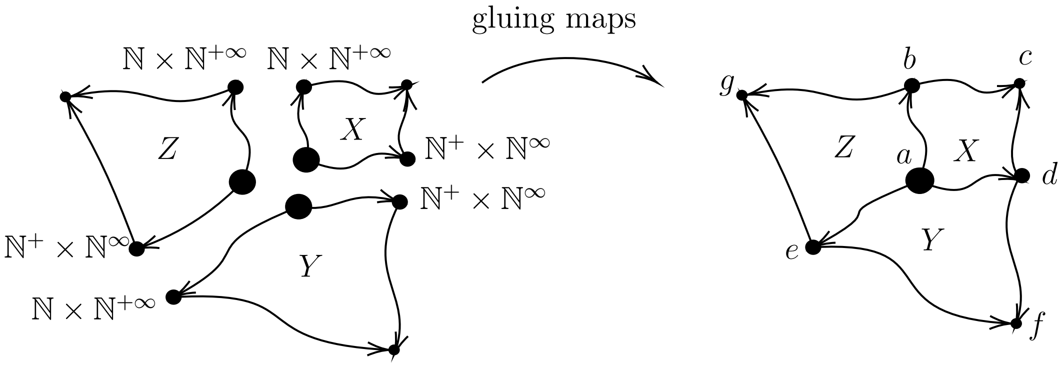


FIGURE 15. Gluing of three $\mathrm{Spec}((\mathbb{N}^2)^\infty)$

In the construction of $\mathbb{P}^2_{\mathbb{F}^1}$, the three copies of $\mathrm{Spec}((\mathbb{N}^2)^\infty)$ are glued along these neighborhoods of $\mathrm{Spec}((\mathbb{N}^2)^\infty)$. We will use the similar notation style as before, $X = \mathrm{Spec}((\mathbb{N}^2)^\infty)$, $Y = \mathrm{Spec}((\mathbb{N}^2)^\infty)$ and $Z = \mathrm{Spec}((\mathbb{N}^2)^\infty)$. Refer to Figure 15.

(1) X and Y are guled along $U_X = \{\{\infty_x\}, \mathbb{N}^+ \times \mathbb{N}^\infty\}$ and $U_Y = \{\{\infty_y\}, \mathbb{N}^+ \times \mathbb{N}^\infty\}$.

(2) X and Z are glued along $U_X = \{\{\infty_x\}, \mathbb{N} \times (\mathbb{N}^+)^\infty\}$ and $U_Z = \{\{\infty_z\}, \mathbb{N} \times (\mathbb{N}^+)^\infty\}$.

(3) Y and Z are glued along $U_Y = \{\{\infty_y\}, \mathbb{N} \times (\mathbb{N}^+)^\infty\}$ and $U_Z = \{\{\infty_z\}, \mathbb{N}^+ \times \mathbb{N}^\infty\}$.

The gluing maps generalize the previous example. They are induced by maps of poinded monoids that are the identity on the $\mathbb{N}^+$ factors and sending the generators of the $\mathbb{Z}$ factors to the negatives of each other in each case. To examine the Hom set from $\mathrm{Spec}(\mathbb{N}^\infty)$ to $\mathbb{P}^2_{\mathbb{F}_1}$, we will use the same convention as before, where f is given by what it does to the topological space and the algebraic $f^\#$ morphism of sheaves we categorize in each case. There will be 19 cases.

Theorem 7.1. $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathbb{P}^2_{\mathbb{F}^1})$ *is a gluing of three* $(\mathbb{N}^\infty)^2$.

Proof.

Subcase 1: 1a2a This is the only f that incorporates all three of our affine copies. The space $\mathrm{Spec}(\mathbb{N}^\infty)$ is sent to the $\{\infty\}$ of X, Y and Z. In this case,

$$U_X = \mathrm{Spec}((\mathbb{Z}^2)^\infty) = \{\{\infty_x\}\}.$$

Likewise, $U_Y = \{\{\infty_y\}\}$ and $U_Z = \{\{\infty\}\}$. When we consider the morphism of sheaves, each generator of each $(\mathbb{Z}^2)^\infty$ is sent to 0. So we get the map $(0,0)$ for X, Y and Z, where the first coordinate indicates where e_1 is being sent to, and the second where e_2 is being sent to. But since we are gluing these points together, the maps are also associated with one another. We notate whens maps are glued with one another by a similarity of pairs $\alpha(\beta, \gamma)$ where α is a local affine chart, β is where e_1 gets sent to and γ is where e_2 gets sent to. For example, $1a2a$ corresponds to the single map of $\mathbb{P}^2_{\mathbb{F}^1}$, $x(0,0) \sim y(0,0) \sim z(0,0)$.

Subcase 2: 1a2d We will avoid some details in the following subcases since we really are just associating maps to $\mathrm{Spec}((\mathbb{N}^2)^\infty)$, where details were given earlier. We do have to be slightly careful about how localization works, however. Here, $\mathrm{Spec}(\mathbb{N}^\infty)$ must be sent to $U_X = \mathrm{Spec}(\mathbb{N}^+ \times \mathbb{Z}^\infty)$ and $U_Y = \mathrm{Spec}(\mathbb{N}^+ \times \mathbb{Z}^\infty)$ in a way that is compatible with the transition function from U_X to U_Y. As described above, this transition function is determined by the map $\mathbb{N}^+ \times \mathbb{Z}^\infty \longrightarrow \mathbb{N}^+ \times \mathbb{Z}^\infty$ that sends $(n, m) \mapsto (n, -m)$ for all n, m. The morphisms $\mathrm{Spec}((\mathbb{N}^2)^\infty) \to U_X$ and $\mathrm{Spec}((\mathbb{N}^2)^\infty) \to U_Y$ both correspond to morphisms of pointed monoids $\mathbb{N}^+ \times \mathbb{Z}^\infty \to \mathbb{N}^\infty$. Moreover, the preimage of ∞ must be ∞ and the preimage of $\mathbb{N}^+ \cup \{\infty\}$ must be $\mathbb{N}^+ \times \mathbb{Z}^\infty$ in both. Thus both must send their generator of the $\mathbb{Z}$ factor to 0 since it does not maps to infinity and 0 is the only unit in $\mathbb{N}$. To commute with the transition function, the generator of both factors of $\mathbb{N}^+$ must map to the same value $a \in \mathbb{N}^+$. Thus we get a map $x(a, 0) \sim y(a, 0)$ for each value of a.

Subcase 3: 1a2b Here, $U_Z = \mathrm{Spec}((\mathbb{Z} \times \mathbb{N}^+)^\infty)$ and $U_X = \mathrm{Spec}((\mathbb{Z} \times \mathbb{N}^+)^\infty)$. Then each $\mathbb{Z} \times (\mathbb{N}^+)^\infty$ send their first generator to 0 and their second generator to $b \in \mathbb{N}^+$. This gives the maps $x(0, b) \sim z(0, b)$ for values $b \in \mathbb{N}^+$.

Subcase 4: 1a2e Let $U_Y = \mathrm{Spec}((\mathbb{N}^+ \times \mathbb{Z})^\infty)$ and $U_Z = \mathrm{Spec}((\mathbb{Z} \times \mathbb{N}^+)^\infty)$. The generators of $(\mathbb{Z} \times \mathbb{N}^+)^\infty$ are sent to 0 and b, respectively, as the transition function is similar to those above but switches factors in the products. So we get the maps $z(b, 0) \sim y(0, b)$.

Subcase 5: 1a2c This map is contained completely only in X. We get the maps $x(a, b)$.

Subcase 6: 1a2f Similarly, we get the maps $y(a, b)$.

Subcase 7: 1a2g Similarly, we get the maps $z(a, b)$.

Subcase 8: 1b2c This map is again completely contained only in X. We must have $e_1 \mapsto a$ and $e_2 \mapsto \infty$. We get the maps $x(a, \infty)$.

Subcase 9: 1b2b Here, $U_X = \mathrm{Spec}((\mathbb{Z}^2)^\infty)$ and $U_Z = \mathrm{Spec}((\mathbb{Z}^2)^\infty)$. The generators of both $(\mathbb{Z}^2)^\infty$ behave as $e_1, g_1 \mapsto 0$ and $e_2, g_2 \mapsto \infty$. We get the map $x(0, \infty) \sim z(0, \infty)$.

Subcase 10: 1b2g This is similar to Subcase 8. We get the map $z(a, \infty)$.

Subcase 11: 1c2c This is simply the constant ∞ map, since we are sending $\mathrm{Spec}(\mathbb{N}^\infty)$ to $\mathrm{Spec}((\mathbb{N}^2 - 0)^\infty)$. We get the map $x(\infty, \infty)$.

Subcase 12: 1g2g This is similar to Subcase 11. We get the map $z(\infty, \infty)$.

Subcase 13: 1f2f This is similar to Subcase 11 & 12. We get the map $y(\infty, \infty)$.

Subcase 14: 1d2c This is similar to Subcase 8, except $e_1 \mapsto \infty$ and $e_2 \mapsto b$. We get the map $x(\infty, b)$.

Subcase 15: 1d2d This is similar to Subcase 9. We get the map $x(\infty, 0) \sim y(\infty, 0)$.

Subcase 16: 1d2f Here we have that $e_1 \mapsto \infty$ and $e_2 \mapsto b$. We get the map $y(\infty, b)$.

Subcase 17: 1e2e This is similar to Subcase 9 and 15. We get the map $y(0, \infty) \sim z(\infty, 0)$.

Subcase 18: 1e2g This is similar to Subcase 8. We get the map $z(\infty, b)$.

Subcase 19: 1e2f This is similar to Subcase 8. We get the map $y(a, \infty)$.

As with Section 6, we get a graph (see Figure 16) that relates the gluing of maps for each copy of $\mathrm{Spec}((\mathbb{N}^2)^\infty)$. □

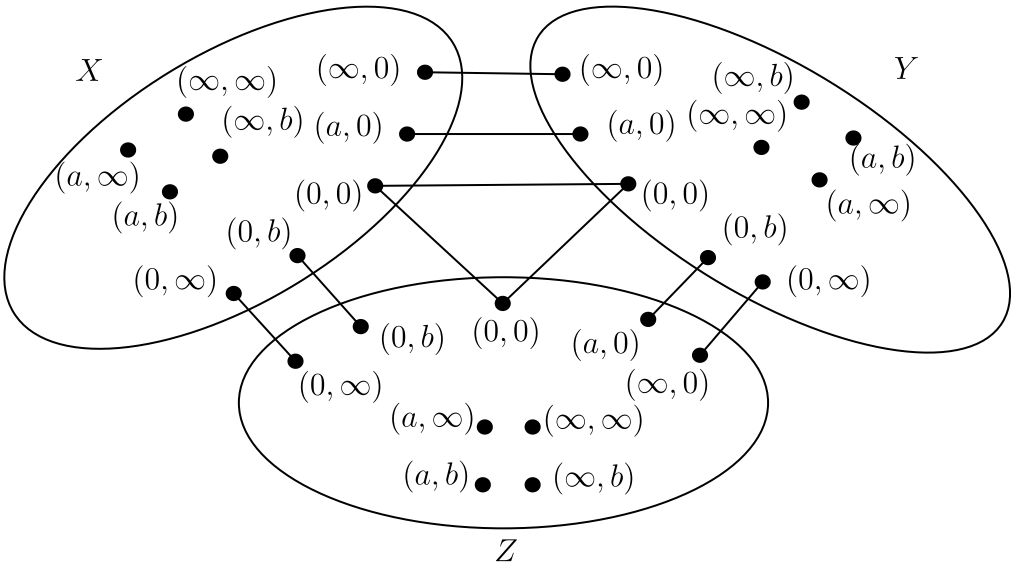


FIGURE 16. Diagram showing gluings of maps in $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathbb{P}^2_{\mathbb{F}^1})$

Theorem 7.2. *The hom sets* $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathbb{P}^2_{\mathbb{F}^1})$ *and* $\mathrm{Hom}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\Delta}_{\mathbb{N}^{2\infty}})$ *are bijective.*

Since we only rigorously introduced $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathbb{P}^2_{\mathbb{F}^1})$, we will characterize $\mathrm{Hom}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\Delta}_{\mathbb{N}^{2\infty}})$ as we prove Theorem 7.1. We employ the methods used in Section 5 as well as the theory discussed in Section 3.

Proof. We will have 7 cases, one for each cone that we can quotient out by. The cones we have are $0, \tau_1, \tau_2, \tau_3, \sigma_1, \sigma_2$ and σ_3. We further notate the faces of each cone. Let the faces of σ_1 be $\tau_{1_1} = \tau_1$ and $\tau_{1_2} = \tau_2$, the faces of σ_2 be $\tau_{2_2} = \tau_2$ and $\tau_{2_1} = \tau_3$ and the faces of σ_3 be $\tau_{3_1} = \tau_1$ and $\tau_{3_2} = \tau_2$. So we have $\tau_{1_2} = \tau_{2_2} = \tau_2$, $\tau_{1_1} = \tau_{3_1} = \tau_1$ and $\tau_{2_1} = \tau_{3_2} = \tau_3$. We adopt this to further hone in that each cone is glued along

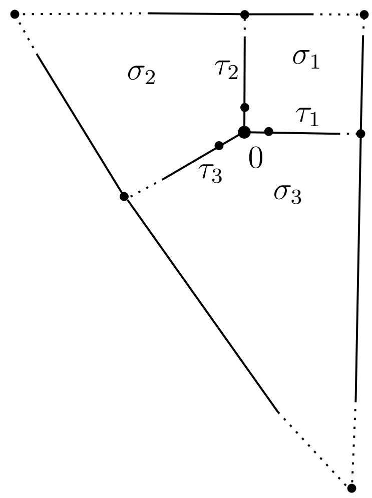


FIGURE 17. Diagram of $\overline{\Delta}_{\mathbb{P}^2_{\mathbb{F}1}}$

the face of another cone.

Case 1: Star(Δ, τ_1) corresponds to the gluing of $F(\sigma_1, \tau_{1_1})$ and $F(\sigma_3, \tau_{3_1})$. By the structure lemma of cones, we get the morphisms $\alpha : \overline{\sigma}_{\mathbb{N}^\infty} \to \overline{F}(\sigma_1, \tau_{1_1})$ or $\beta : \overline{\sigma}_{\mathbb{N}^\infty} \to \overline{F}(\sigma_3, \tau_{3_1})$. To preserve the vector space lattice, 1 in $\overline{\sigma}_{\mathbb{N}^\infty}$ must be sent to a nonnegative integer in F(σ_1, τ_{1_1}) or F(σ_3, τ_{3_1}). If 1 is sent to 0 in both cones, these maps are the same which makes sense since these cones only intersect at their 0's. Otherwise, α sends 1 to $a_\alpha \in \mathbb{N}^+$ or β sends 1 to $b_\beta \in \mathbb{N}^+$. We must be careful in that the $\mathbb{N}^+$ in each of the $\overline{F}(\sigma_1, \tau_{1_1})$ and $\overline{F}(\sigma_3, \tau_{3_1})$ are distinct. Thus we get a fan that is isomorphic to $\overline{\Delta}_{\mathbb{P}^1_{\mathbb{F}1}}$. So the possible maps we get are all of $\sigma_1(\infty, 0) \sim \sigma_3(\infty, 0)$, $\sigma_1(\infty, b)$ and $\sigma_3(\infty, b)$, using a similar notation as in the previous proof.

Case 2: Star(Δ, τ_2) corresponds to the gluing of $\overline{F}(\sigma_1, \tau_{1_2})$ and $\overline{F}(\sigma_2, \tau_{2_2})$. This case similarly decomposes like Case 1. We get the following maps: $\sigma_1(0, \infty) \sim \sigma_2(0, \infty)$, $\sigma_1(a, \infty)$ and $\sigma_2(a, \infty)$.

Case 3: Star(Δ, τ_3) corresponds to the gluing of $\overline{F}(\sigma_3, \tau_{3_2})$ and $\overline{F}(\sigma_2, \tau_{2_1})$. This is similar to Cases 1 and 2. The maps we get are $\sigma_2(\infty, 0) \sim \sigma_3(0, \infty)$, $\sigma_2(\infty, b)$ and $\sigma_3(a, \infty)$.

Case 4,5,6: Case 4,5 and 6 are all similar in that we will be quotienting out by σ_i for $i = 1, 2, 3$. Here we are modding out by the interior of each σ_i in $\overline{\Delta}$. So we get the constant maps $\sigma_1(\infty, \infty)$, $\sigma_2(\infty, \infty)$ and $\sigma_3(\infty, \infty)$ into each of the three extremal vertices of the triangle.

Case 7: This case is by far the most interesting one, as it will give us the most maps. For each cone σ_i, $\overline{\sigma}_{\mathbb{N}^\infty}$ is sending 1 to $F(\sigma_i, 0)$. As we have seen earlier in the paper, this means 1 is sent to any integer pair $(a, b) \in \mathbb{N}^2$. Let us remind ourselves how each σ_i is glued with one another. The cones σ_1 and σ_2 are glued along σ_1's and σ_2's 'y-axis', σ_1 and σ_3 are glued on σ_1's and σ_2's 'x-axis' and σ_2 and σ_3 are glued along σ_2's 'x-axis' and σ_3's 'y-axis'. Thus the only way maps from different cones may potentially be glued with one another is along these faces of each cone. The only

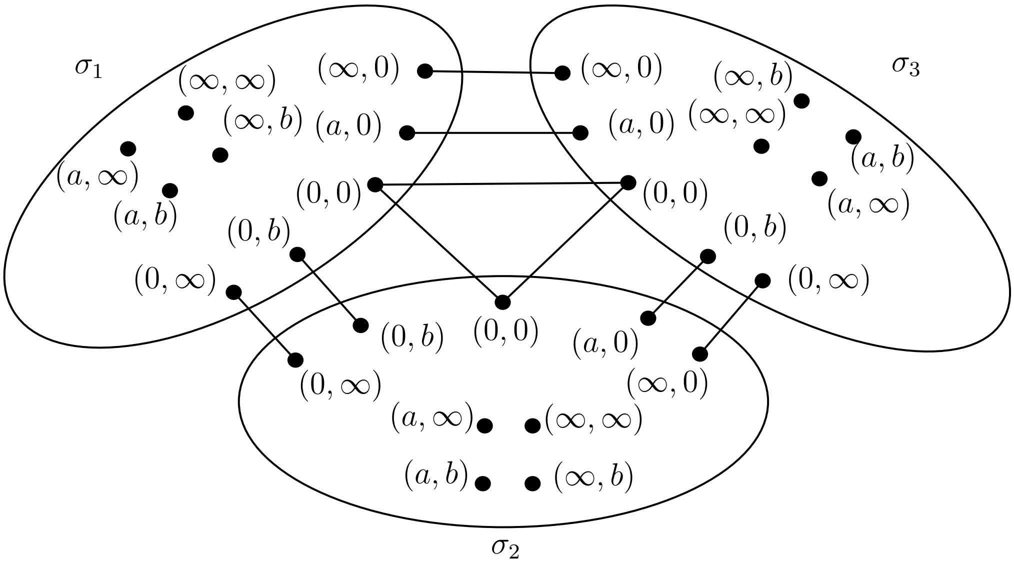


FIGURE 18. Diagram showing gluings of maps in $\mathrm{Hom}(\overline{\sigma}_{\mathbb{N}^\infty}, \overline{\Delta}_{\mathbb{P}^2_{\mathbb{F}^1}})$

map that lands completely in all three maximal cones is the constant map $1 \mapsto 0$ to the internal origin. We get from this the map $\sigma_1(0,0) \sim \sigma_2(0,0) \sim \sigma_3(0,0)$. The maps that are glued along σ_1 and σ_3 are $\sigma_1(a,0) \sim \sigma_3(a,0)$. The maps that are glued along σ_1 and σ_2 are $\sigma_1(0,b) \sim \sigma_2(0,b)$. The maps that are glued along σ_2 and σ_3 are $\sigma_2(a,0) \sim \sigma_3(0,b)$. Thus, those are the only maps we get from $F(\sigma_i, 0)$ that are glued with some $F(\sigma_j, 0)$. What remains are the $(a,b) \in (\mathbb{N}^2)^+$ in each cone, so we get maps $\sigma_1(a,b), \sigma_2(a,b)$ and $\sigma_3(a,b)$ for each $a, b \in \mathbb{N}^+$. Thus we get the graph in Figure 18, which is the same diagram for $\mathrm{Hom}(\mathrm{Spec}(\mathbb{N}^\infty), \mathbb{P}^2_{\mathbb{F}^1})$. □

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