Differential Equations Dictionary

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Summer 2020

The aim of this paper is to provide an organized (and hopefully coherent) presentation of methods and techniques covered in MAT 411, Summer 2020. As advice on how to use this: I wrote this document linearly so I will reference myself from earlier. I only cover the barebone structure of the techniques and don't do any examples, although doing examples is how you will get more comfortable with using these methods. I do my best to try to imitate the computational hurdles needed to follow through each technique.

These are solely my personal notes and do not give insight into any exam questions. Please let me know of mistakes.

1 Linear First Order Equations

These are equations that are of the form

$$y' + P(t)y = Q(t) \tag{1}$$

where P and Q are functions of t.

1.1 Separable Equations

This is the easiest case of differential equations. If we can express the differential equation as

$$y' = g(t)h(y) \tag{2}$$

then diving by $h(y)$ we may integrate

$$\int \frac{1}{h(y)}dy = \int g(t)dt$$

Let $H(y)$ be the antiderivative of $1/h(y)$ and $G(t)$ be an antiderivative of $g(t)$. Then the solution to (2) is

$$H(y) = G(t) + C$$

1.2 Integrating Factor

The differential equation (1) is in *standard form*: coefficient of y' is 1. The technique of using an integrating factor means we would like to make the left hand side more digestible by multiplying *something*. Let the integrating factor be this *something*

$$\mu(t) = \exp\left(\int P(t)dt\right)$$

Multiply equation (1) by $\mu(t)$ and we can decompress the left hand side into

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)Q(t) \tag{3}$$

When integrating (3) over t, the integral and derivative will cancel out by the fundamental theorem of calculus. This leaves

$$\mu(t)y = \int \mu(t)Q(t)dt \quad \Rightarrow \quad y = \frac{1}{\mu(t)} \int \mu(t)Q(t)dt$$

as a solution to (1).

1.3 Exact Equations

Let the expression $M(x,y)dx + N(x,y)dy$ be called a *differential form*. A differential form is said to be *exact* if $M_y = N_x$. Remark: Really there should be some notion of a neighborhood where this differential form is defined, but in computations this is usually overlooked. An exact differential form gives an exact differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{4}$$

which is the differential equation we are interested in solving. We are going to construct a solution $F(x,y)$ to this differential equationwith the property that $F_x = M$ and $F_y = N$. Recall by Clairaut's theorem, the order of partial differentiation can be swapped. Begin with integrating the equation $F_x = M$ with respect to x

$$F(x,y) = \int M(x,y)dx + g(y) \tag{5}$$

where the $g(y)$ pops up since $\frac{d}{dx}g(y) = 0$. Let $L(x,y)$ be the antiderivative of $M(x,y)$. What is left to do is to find what $g(y)$ is (I interpret $g(y)$ as a "correction factor" to $L(x,y)$, you should understand what I mean by the end of this derivation). Deriving (5) with respect to y will give

$$\begin{aligned} N(x,y) &= \frac{d}{dy}L(x,y) + g'(y) \\ N(x,y) - \frac{d}{dy}L(x,y) &= g'(y) \end{aligned} \tag{6}$$

Then integrate (6) with respect to y (by now we should know explicitly what $\frac{d}{dy}L(x,y)$ is) to obtain $g(y)$ with a constant of integration C attached. Then we obtain

$$F(x,y) = L(x,y) + g(y) = C$$

to be the solution to (4).

This technique follows (somewhat cleanly) if equation (4) is exact. But what happens if we *don't* have exactness? Then we employ a *special* integrating factor in the following fashion.

$$\text{If} \quad \frac{M_y - N_x}{N} \text{ is a function of } x, \text{ then let } \mu(x) = \exp\left(\frac{M_y - N_x}{N}\right)$$

$$\text{If} \quad \frac{N_x - M_y}{M} \text{ is a function of } y, \text{ then let } \mu(y) = \exp\left(\frac{N_x - M_y}{M}\right)$$

Be careful that tweeking & algebra may be required to cancel out expressions and obtain a single variable. Once you multiply (4) by $\mu(x)$ or $\mu(y)$, then proceed with the technique described above. By multiplying the equation by $\mu(x)$ or $\mu(y)$, you may "lose" solutions.

1.4 Homogenous Equations

This is our first technique that involves substitutions. Suppose we have a differential equation

$$y' = f(x,y)$$

If $f(x,y) = g(x)h(y)$ for some g and h, then this equation is separable. If $f(x,y)$ is a function given by the ratio y/x, the differential equation is then called *homogenous*. In this case, we use the substitutions

$$v = \frac{y}{x}, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Let $f(x,y) = G(v)$. Then we obtain the equation

$$v + x\frac{dv}{dx} = G(v) \quad \Rightarrow \quad \frac{dv}{G(v) - v} = \frac{dx}{x}$$

Integrate and then substitute y/x for v.

1.5 Equations of form y'=G(ax+by)

These are straight forward. Use the substitution $z = ax + by$.

1.6 Bernoulli Equations

An differential equation is called *Bernoulli* if it has the form

$$y' + P(t)y = Q(t)y^n \tag{7}$$

for some n a real number. This differential equation gets a special name because we can transform (7) into a linear equation by the substitution

$$v = y^{1-n}, \quad \frac{1}{1-n}\frac{dv}{dt} = y^{-n}\frac{dy}{dt}$$

The power of this substitution is noticeable when we multiply (7) by y^{-n}.

$$y^{-n}y' + P(t)y^{1-n} = Q(t) \quad \Rightarrow \quad \frac{1}{1-n}\frac{dv}{dt} + P(t)v = Q(t)$$

An integrating factor will usually finish these up but be sure to have the solution in the original variables.

1.7 Linear Coefficients

These equations adopt a more relaxed appearance than equation (4),

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0 \tag{8}$$

We first consider the case where $a_1b_2 = a_2b_1$. If this is the case, let $z = a_1x + b_1y$ and $z' = a_1 + b_1y'$. Then equation (8) becomes

$$\frac{1}{b_1}(z' - a_1) = -\frac{z + c_1}{\frac{b_1}{b_2}z + c_2}$$

in which we can use technique 1.5 to wrap it up. Secondly, consider the case where $a_1b_2 \neq a_2b_1$ *and* $c_1 = c_2 = 0$. Then we may rewrite equation (8)

$$y' = -\frac{a_1 + b_1(y/x)}{a_2 + b_2(y/x)} \tag{9}$$

and use the substitution to $v = y/x$ to finish this up. The previous substitutions only introduced one new variable to replace y. But now we will transform both the independent variable x and dependent y. The motivation of this substitution is to make equation (8) "look like" equation (9), even if either constants c_1 or c_2 are nonzero. With this in mind, we will introduce the substitutions

$$x = u + h \quad \text{and} \quad y = v + k$$

where the constants h, k will act as "buffers" or "corrections" to make the constants after substituting zero. Substituting gives

$$(a_1u + b_1y + (a_1h + b_1k + c_1))dx + (a_2u + b_2y + (a_2h + b_2k + c_2))dy = 0$$

We would like to choose a particular pair of constants h and k such that

$$a_1h + b_1k + c_1 = 0$$

$$a_2h + b_2k + c_2 = 0$$

are simultaneously zero. In other words, we want the solution to

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} -c_1 \\ -c_2 \end{pmatrix}$$

Notice that the condition $a_1b_2 \neq b_1a_2$ in linear algebra terms means the determinant of this matrix is nonzero, hence we indeed have the existence of a unique solution. Solving for h and k will give the differential equation

$$\frac{dv}{du} = -\frac{a_1 + b_1(v/u)}{a_2 + b_2(v/u)}$$

which does indeed look like equation (9).

2 Linear Second Order Equations

We now move onto equations of order 2. These are equations that are of the form

$$ay'' + by' + cy = f(t) \tag{10}$$

where second order means $a \neq 0$.

2.1 Homogenous Equations

This is the simplest case of second order equations. This is where $f(t) = 0$ in equation (10) and we call these equations *homogenous*. The substitution we will use is $y = e^{rt}$ so that (10) becomes

$$e^{rt}(ar^2 + br + c) = 0$$

Since e^{rt} is never zero for any t, we may divide by e^{rt} and what we are left with is called the *auxiliary equation* associated to the homogenous equation. We may apply the quadratic formula to $r^2 + br + c$ to find suitable roots r_1 and r_2. There are three different cases for these roots.

Case 1 r_1 and r_2 are real and distinct. This gives that both $y_1(t) =^{r_1t}_e$ and $y_2(t) = e^{r_2t}$ are roots to the homogenous equation. Linear combinations of $y_1(t)$ and $y_2(t)$ form the family of solutions to the homogenous equation. Therefore our general solution is

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}, \quad c_1, c_2 \in \mathbb{R}$$

Case 2 $r_1 = r_2$. It is clear that $y_1(t) = e^{r_1}$ should be one solution and certainly $y_2(t) = e^{r_2}$ is just a relabeling of y_1. The second-order-ness of the equation suggests there should be a second linearly independent solution. One way to obtain this solution is to use *reduction of order* which I won't mention until later. The *real* second solution is $y_2(t) = te^{r_1}t$ to give

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} \quad c_1, c_2 \in \mathbb{R}$$

You should check that this indeed is a solution to the homogenous equation when the auxiliary equation gives only one root.

Case 3 r_1 and r_2 are complex conjugates. To *fully* understand whats going on in the background, one should recall Euler's formula which says $e^{i\theta} = \cos\theta + i\sin\theta$. The root r_1 takes the form $\alpha + i\beta$, where r_2 takes the form $\alpha - i\beta$. Then the solutions we obtain are $e^{(\alpha+i\beta)t}$ and $e^{(\alpha-i\beta)t}$. Euler's' formula implies that for either of these solutions, we must have that the real *and* imaginary parts of $e^{(\alpha+i\beta)t}$ are *also* solutions to the homogenous equation. Thus the general solution takes the form

$$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t, \quad c_1, c_2 \in \mathbb{R}$$

2.2 Nonhomogenous Equations

We now consider the monumental task of trying to classify the solutions of

$$ay'' + by' + cy = f(t) \tag{11}$$

where $f(t)$ may not necessarily be identically zero. Here we will employ **two** classes of differential equations and give a corresponding *particular* solution (here particular means we will only muster up a solution that does not encompass every other solution). Suppose our first class of equations takes the form

$$ay'' + by' + cy = Ct^m e^{rt}$$

where $m \in \mathbb{Z}^{\geq 0}$. Even though the RHS may not be identically zero, we will consider the *associated homogenous equation*, $ay'' + by' + cy = 0$. Just as in 2.1, this gives an associated auxiliary equation. A particular solution is

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt},$$

where s denotes the multiplicity of r in the associated auxiliary equation $ar^2 + br + c = 0$. To solve for $A_m, ..., A_0$, we compute the first and second derivatives of y_p, substitute y, y', y'' for y_p, y_p', y_p''' respectively into $ay'' + by' + cy = Ct^m e^{rt}$ and solve for the constants. As a disclaimer the arithmetic for this procedure is often miserable, so be very careful and take your time.

Suppose the second class of equationsi takes the form

$$ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t}\cos\beta t \\ Ct^m e^{\alpha t}\sin\beta t \end{cases}$$

where $\beta \neq 0$. Use the particular solution

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t + t^s(B_m t^m + \cdots + B_1 t + B_)e^{\alpha t}\sin\beta t,$$

Use the same procedure as above to solve for what the constants $A_m, ..., A_0, B_m, ..., B_0$ could be. As you could imagine, the arithmetic for this one is even worse.

After covering these classes, we now move up to the general solution of the nonhomogenous equation (11) when we already have a particular solution. These combine what was just introduced and homogenous solutions. For a nonhomogenous equation (11), a general solution is

$$y(t) = y_p(t) + c_1 y_1(t) + c_2 y_2(t) \tag{12}$$

where $y_1(t), y_2(t)$ are solutions to the associated homogenous equation. Using linearity, one could show that (12) is indeed a solution to (11).

2.3 Variation of Parameters

Sections 2.1 and 2.2 cover preliminary methods on how to tango with second order equations. Here, we'll introduce cleaner and more general machinery to produce a particular solution. Suppose we have the usual suspect

$$ay'' + by' + cy = f(t) \tag{13}$$

For the associated homogenous equation, we obtain the general solution $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$ where the h emphasizes the equation being homogenous. What we do is *replace* the constant coefficients c_1 and c_2 with $v_1(t)$ and $v_2(t)$ to give the function $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, where p denotes the particular-ness and 'elevation' of our scope from homogenous to nonhomogenous. We will impose **three** conditions on the functions $y_p(t), v_1$ and v_2, one obvious and the other two not so obvious.

(i) The obvious condition is we want $y_p(t)$ to satisfy equation (13)

$$ay_p'' + by_p' + cy_p = f(t)$$

after all we are looking for solutions to (13).

(ii) The second condition is to require

$$v_1'y_1 + v_2'y_2 = 0$$

The reasoning because if we get rid of the terms that involve the first derivative of v_1 and v_2, then we won't have to care about the second derivatives of v_1 and v_2. We would presumably obtain a larger family of solutions to (13) but we are not that ambitious...yet.

Imposing these conditions give us the following set up

$$\begin{aligned}
y_p &= v_1 y_1 + v_2 y_2 \\
y_p' &= v_1 y_1' + v_2 y_2' \\
y_p'' &= v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''
\end{aligned}$$

Plugging in these values into (13) spit out

$$\begin{aligned}
f &= ay_p'' + by_p' + cy_p \\
&= a(v_1'y_1' + v_1 y_1'' + v_2'y_2' + v_2 y_2'') + b(v_1 y_1' + v_2 y_2') + c(v_1 y_1 + v_2 y_2) \\
&= a(v_1'y_1' + v_2'y_2') + v_1(ay_1'' + by_1' + cy_1) + v_2(ay_2'' + by_2' + cy_2) \\
&= a(v_1'y_1' + v_2'y_2')
\end{aligned}$$

where the last two terms in the second to last equality vanish since we first figured out that y_1 and y_2 were solutions to the associated homogenous equation.

(iii) The last condition we impose is

$$y_1'v_1' + y_2'v_2' = \frac{f}{a}$$

It obvious here that we need $a \neq 0$, or else we'd drop to the first order case.

In practice, we really only focus on conditions (ii) and (iii) to sew together the particular solution as our final answer. I will elaborate more on what to do once you come to (ii) and (iii). At this point, we know what y_1 and y_2 and we are looking for v_1' and v_2'. Condition (ii) implies

$$v_1' = \frac{-y_2 v_2'}{y_1}$$

and substituting this into condition (iii) gives

$$\begin{aligned}\frac{f}{a} &= y_1'\left(\frac{-y_2 v_2'}{y_1}\right) + v_2' y_2' \\ &= v_2'\left(y_2' - \frac{y_1' y_2}{y_1}\right)\end{aligned}$$

Then we isolate the v_2' and integrate with respect to t to obtain v_2 (don't forget the $+c_2$!) Use condition (ii) and integrate to obtain v_1 (don't forget the $+c_2$!!). Once you have the functions v_1 and v_2, substitute them back into y_p for your particular solution.

2.4 Existence & Uniqueness and IVP's

Up until now, I've neglected EU and IVP's so I will mention it here in the case for nonhomogenous second order differential equations with variable coefficients. We have not discussed variable coefficients yet, only constant coefficients so consider this is our first impressions with them.

Theorem *Existence and Uniqueness of Solutions* Suppose we are given continuous functions $p(t), q(t)$ and $g(t)$ on (a, b) that contains a point t_0. Then for any initial values $Y_0, Y_1 \in \mathbb{R}$, there exists a unique solution $y(t)$ on (a, b) to the initial value problem

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t), \quad y(t_0) = Y_0,\ y'(t_0) = Y_1$$

While having a theorem that tells us whether or not there exists a unique solution to an IVP is really powerful, the theorem does not tell us *how* to construct the solution in a general sense.

2.5 Variable Coefficient Equations

Here we begin with a simple first swing at linear second order equation's with variable coefficients. Suppose we have the equation

$$at^2 y''(t) + bty'(t) + cy = f(t) \tag{14}$$

for $a, b, c \in \mathbb{R}$. The equations of the form (14) are called *Cauchy-Euler*. We use the simple substitution $y = t^r$. Assuming $t > 0$, we substitute to transform (14) into

$$t^r(ar^2 + (b-a)r + c) = 0 \Rightarrow ar^2 + (b-a)r + c = 0$$

We call this the *associated characteristic equation.* To solve for r, we use the quadratic formula and unsurprisingly there are three cases.

Case 1 Roots r_1 and r_2 are real and distinct. Then this produces two linearly independent solutions

$$y_1(t) = t^{r_1}, \qquad y_2(t) = t^{r_2}$$

Case 2 Roots are equal. Then this produces the two linearly independent solutions

$$y_1(t) = t^r, \qquad y_2(t) = t^r \ln(t)$$

Case 3 Roots are complex conjugates. Let $r = \alpha + i\beta$. Then

$$t^r = e^{r\ln(t)} = t^\alpha e^{i\beta \ln t} = t^\alpha(\cos\beta\ln t + i\sin\beta\ln t)$$

As in Section 2.1, we obtain the two linearly independent solutions

$$y_1(t) = t^\alpha \cos\beta\ln t, \quad y_2(t) = t^\alpha \sin\beta\ln t$$

2.6 A Note on Linear Independence

Something that should be in the back of our minds is if solutions y_1 and y_2 for a differential equation are linearly dependent. If they are linearly dependent, then we *really* only have one solution.

A quick (and stripped down) way to verify if y_1 and y_2 are linearly independent for a homogenous equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0 \tag{15}$$

is to compute the *Wronksian*. Let I be an interval where $p(t)$ and $q(t)$ are simultaneously continuous. If the Wronkskian

$$\det\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = 0$$

for any $t \in I$, then y_1 and y_2 are linearly dependent. We could have also formulated the statement in terms of the contrapositive if you'd prefer that.

Alternatively if you have a non-zero solution to (13), call it $y_1(t)$, and you would like to produce a second solution y_2 linearly independent to y_1, use the *reduction of order* formula

$$y_2(t) = y_1(t)\int \frac{e^{-\int p(t)dt}}{y_1(t)^2}dt$$