Waldschmidt Constant of Complex Reflection Groups

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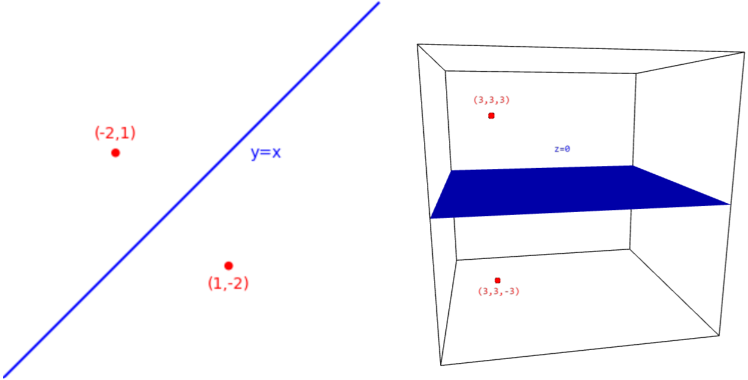
A **pseudo-reflection** is a linear map $r : V \to V$ that fixes a hyperplane pointwise and has finite order.



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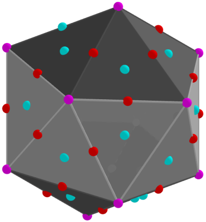
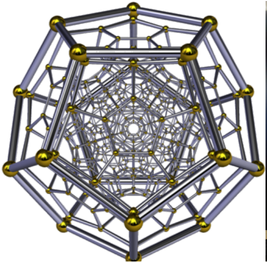
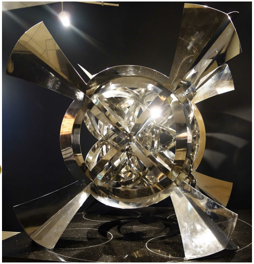
Definition

A linear group $G \subseteq \mathrm{GL}(V)$ is a **complex reflection group** if

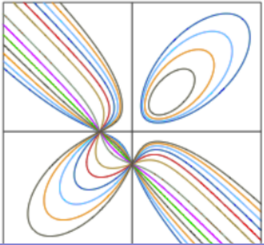
- V is a vector space over $\mathbb{C}$
- G is finite, and
- G is generated by pseudo-reflections.

Where do complex reflection groups exist?

- Shapes

- Pencil of curves



The icosahedral symmetry group $G = A_5 \times \mathbb{Z}_2$

The group G has order $|G| = 120$ and 15 reflections. The group G is generated by the reflections

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \frac{-1}{2} \begin{pmatrix} \phi - 1 & \phi & 1 \\ \phi & -1 & \phi - 1 \\ 1 & \phi - 1 & -\phi \end{pmatrix}.$$

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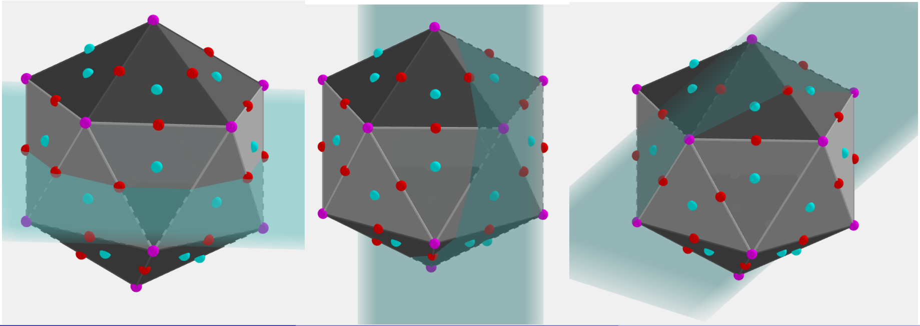

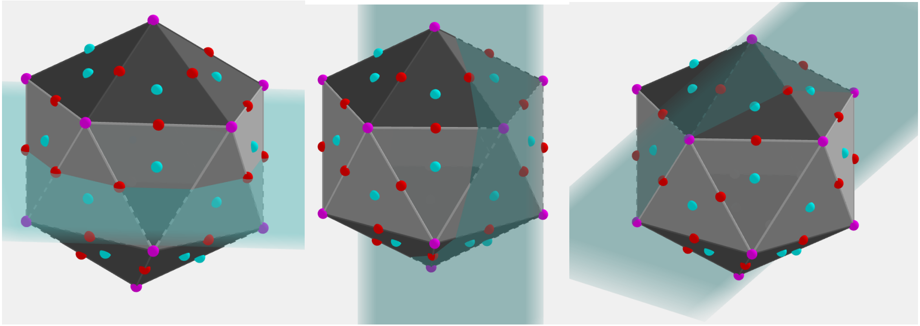

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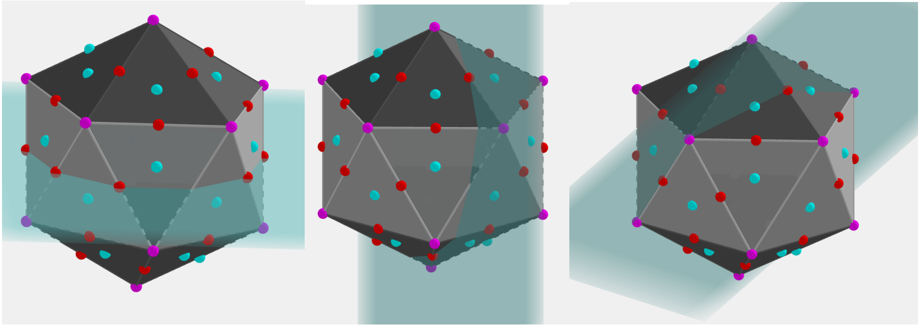

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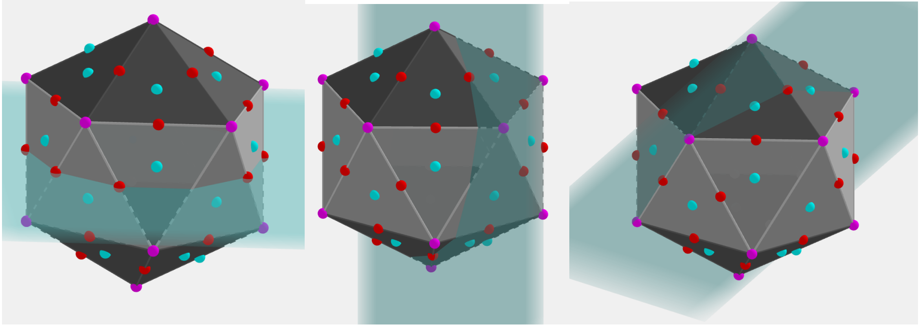

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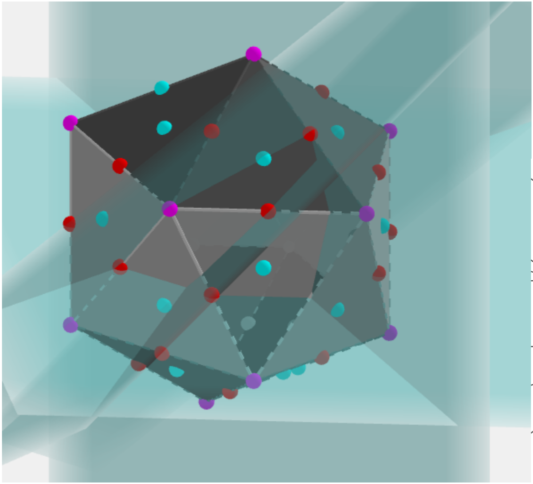
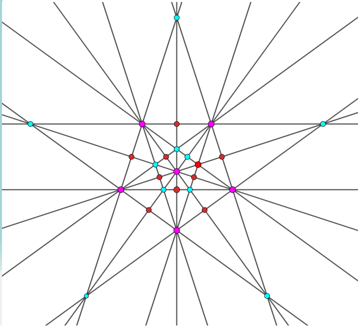
There are 15 reflecting hyperplanes, each with its defining polynomial. Take ψ_{15} to be the product of the 15 defining polynomials.

After projectivization to $\mathbb{P}^2$, $\psi_{15} = 0$ defines a **line configuration** $\mathcal{A}$.





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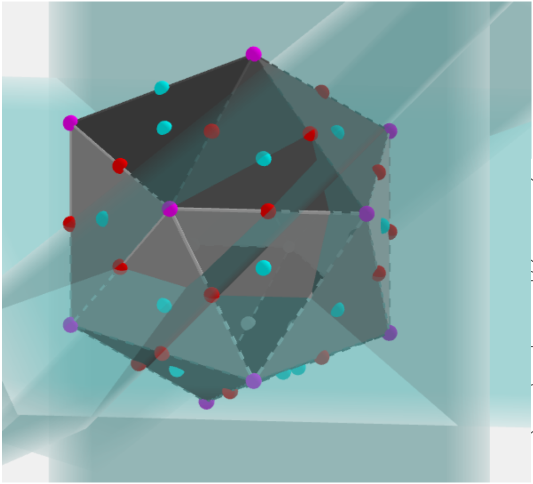
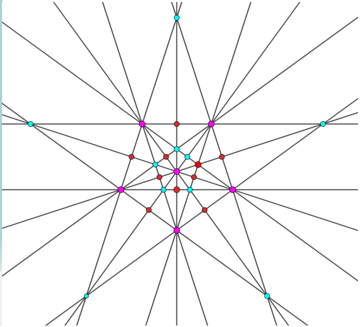
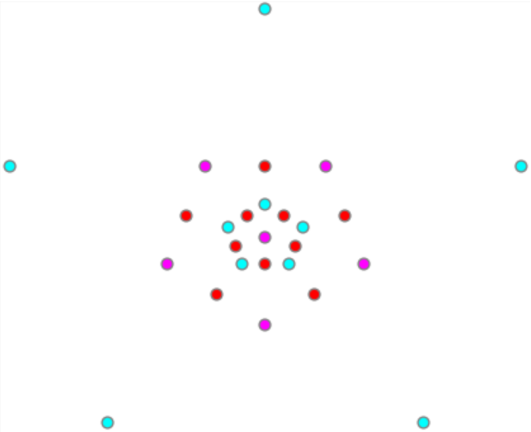




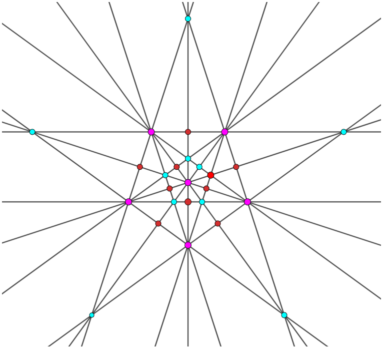
There are 15 double points, 10 triple points, and 6 quintuple points.



Question

For fixed $m \in \mathbb{Z}^+$, what is the minimal degree d of a curve that passes through each of the $31(= 15 + 10 + 6)$ singularities of $\mathcal{A}$ with multiplicity at least m?

The defining polynomial ψ_{15} of degree $d = 15$ of the line configuration $\mathcal{A}$ has multiplicity at least $m = 2$ at each singularity.



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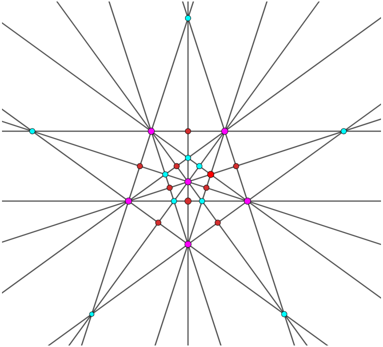

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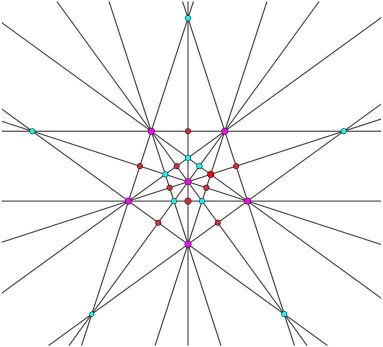

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Theorem (C '24)

If C is a curve of degree d having multiplicity at least m at the 31 singularities of $\mathcal{A}$, then $\frac{d}{m} \geq \frac{11}{2}$.

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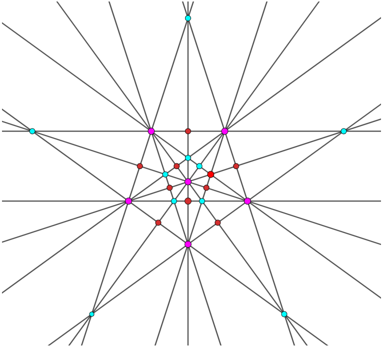

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If C is a curve of degree d having multiplicity at least m at the 31 singularities of $\mathcal{A}$, then $\frac{d}{m} \geq \frac{11}{2}$.This is sharp: there exists a curve of degree 66 with multiplicity 12 at the singularities.

Algebraic notation

Points and ideals

For each singularity $p \in \mathcal{A}$, we have an ideal $I_p \subseteq \mathbb{C}[x, y, z]$. For example,

$$p = [1 : 1 : 1] \longleftrightarrow I_p = \langle x - z, y - z \rangle$$

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Multiplicity

For a singularity $p \in \mathcal{A}$, the m-th power of an ideal $I_p^m \subseteq \mathbb{C}[x, y, z]$ is the set of curves passing through p with multiplicity m.

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- $d = \alpha(I^{(m)})$.

The **Waldschmidt constant** of I is defined to be

$$\widehat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m}$$

Question (Restated)

Compute $\widehat{\alpha}(I_{\mathcal{A}})$, where $I_{\mathcal{A}}$ is the defining ideal of the 31 points of $\mathcal{A}$.

Ring of G-invariant polynomials

The ring is a $\mathbb{C}$-algebra generated by ψ_2, ψ_6, and ψ_{10}.

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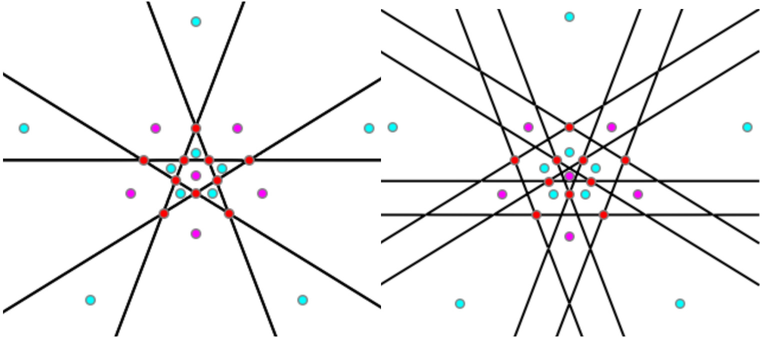

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passes through each double and triple point with multiplicity 6 and each quintuple point with multiplicity 2.

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A particular curve

The curve D given by
$$\psi_{15}^2\psi_6\psi_{30} = 0$$
has degree 66 and multiplicity at least 12 at each singularity.

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Proof

- We can construct a **family of curves** D_k of degree $55k + 2$ with multiplicity $10k$ at each of the 31 points.

$$\widehat{\alpha}(I_{\mathcal{A}}) = \lim_{m\to\infty} \frac{\alpha(I_{\mathcal{A}}^{(m)})}{m} \leq \lim_{k\to\infty} \frac{55k+2}{10k} = \frac{11}{2}.$$

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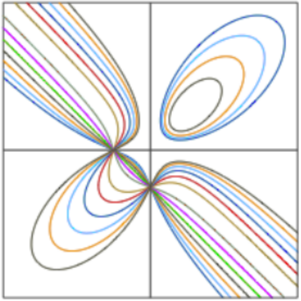
- If an irreducible curve of degree d with multiplicity m at the 31 points has $\frac{d}{m} < \frac{11}{2}$, then the curve must be an irreducible component of D. But none of the irreducible components of ψ_{15}, ψ_6 and ψ_{30} satisfy $\frac{d}{m} < \frac{11}{2}$.

The Hessian group $G = \mathrm{ASL}(2,3) \times \mathbb{Z}_3$

The Hesse pencil is the one-dimensional linear system of plane cubic curves given by

$$\lambda(x^3 + y^3 + z^3) + \mu xyz = 0, \qquad [\lambda : \mu] \in \mathbb{P}^1.$$



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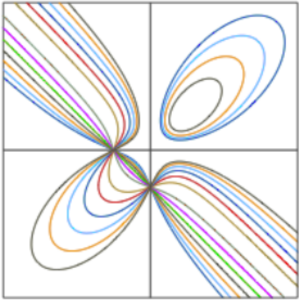

The Hessian group G is the group that preserves the Hesse pencil.

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Let $I_{\mathcal{B}}$ be the defining ideal of 21 singularities of $\mathcal{B}$. Then

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- If an irreducible curve of degree d with multiplicity at least m at the 21 points has $\frac{d}{m} < \frac{9}{2}$, then the curve must be an irreducible component of $\mathcal{C}$ but the curve $\mathcal{C}$ is already irreducible and has $\frac{d}{m} > \frac{9}{2}$.

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- There is a family of curves of degree $36k$ with multiplicity $8k$ at the 21 points.

$$\widehat{\alpha}(I_{\mathcal{B}}) = \lim_{m\to\infty} \frac{\alpha(I_{\mathcal{B}}^{(m)})}{m} \leq \lim_{k\to\infty} \frac{36k}{8k} = \frac{9}{2}.$$

Containment Problem

When are $I^{(n)}$ and I^n? equal?

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Example of $I^{(2)} \not\subseteq I^2$

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- But $\alpha(I^2) = 4$.
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Resurgence

The **resurgence** of an ideal I is defined to be

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Bounds

By Ein-Lazarsfeld-Smith, we have the bound $1 < \rho(I) \leq 2$.

- $I^{(4)} \subseteq I^2$.

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Let $I_{\mathcal{A}}$ be the defining ideal of the 31 singularities of $\mathcal{A}$. Then

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Theorem (C '24)

Let $I_{\mathcal{B}}$ be the defining ideal of the 21 singularities of $\mathcal{B}$. Then

$$\rho(I_{\mathcal{B}}) = 8/7.$$

Proof (of $\rho(I_{\mathcal{A}}) = 9/8$).

- We can show the failure of containment $I^{(9)} \not\subseteq I^8$.
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- Bocci-Harbourne says the failure of containment $I_{\mathcal{A}}^{(m)} \not\subseteq I_{\mathcal{A}}^r$ implies the last inequality

$$\frac{11}{2}m = m\widehat{\alpha}(I_{\mathcal{A}}) \leq \alpha(I_{\mathcal{A}}^{(m)}) \leq 6r + 4.$$

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- Finitely many pairs (m, r) satisfy this system.

Proof cont.

```
i18 : for m from 1 to 24 do ( for r from 1 to floor( 32/3 * m / 12 ) do (
      if 11*m - 8 <= 12 * r  then (
      << "Containment is " << isSubset(symI(m),I^r) << " for m = "<< m << ", r = "<<  r << endl << flush););
Containment is true for m = 4, r = 3
Containment is true for m = 5, r = 4
Containment is true for m = 6, r = 5
Containment is true for m = 7, r = 6
Containment is true for m = 8, r = 7
Containment is false for m = 9, r = 8
Containment is true for m = 16, r = 14
Containment is true for m = 17, r = 15
Containment is true for m = 18, r = 16
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- But the finitely many containments actually do hold except for $m = 9$ and $r = 8$.
- Therefore no such $\frac{m}{r} > \frac{9}{8}$ exists such that $I_{\mathcal{A}}^{(m)} \not\subseteq I_{\mathcal{A}}^{r}$ and

$$\rho(I_{\mathcal{A}}) = 9/8.$$

Thank you!