Polynomial Interpolation in Algebraic Geometry

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Polynomial Interpolation

I. Single variable polynomial interpolation

Example 1

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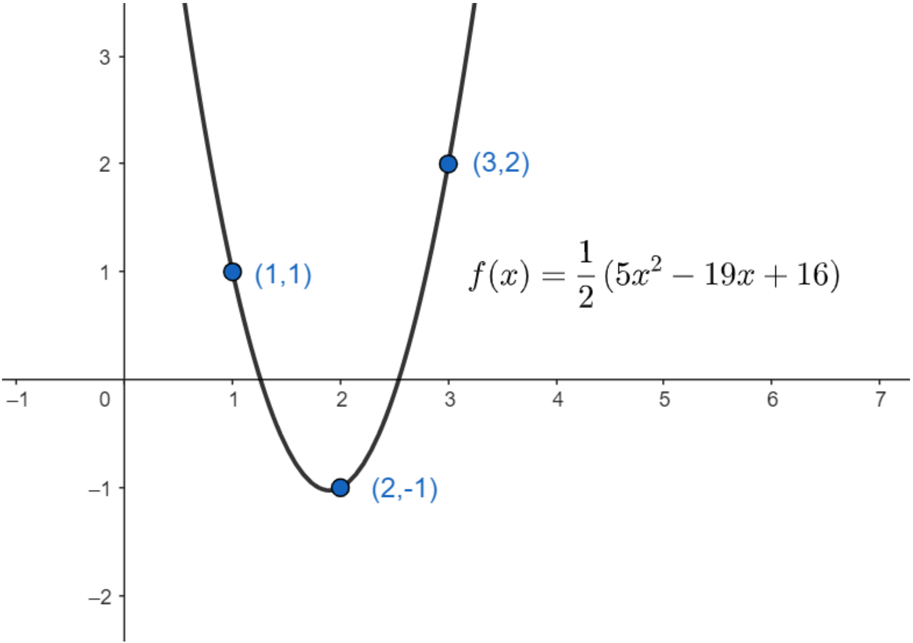
$$f(2) = 1 \cdot \frac{(2-2)(2-3)}{(1-2)(1-3)} - 1 \cdot \frac{(2-1)(2-3)}{(2-1)(2-3)} + 2 \cdot \frac{(2-1)(2-2)}{(3-1)(3-2)} = -1$$



(3,2)
(1,1)
(2,-1)
$f(x) = \frac{1}{2}(5x^2 - 19x + 16)$

Lagrangian interpolation

We are given

$$x_1, \ldots, x_n \text{ in } \mathbb{R} \text{ distinct } \textbf{points}$$
$$y_1, \ldots, y_n \text{ in } \mathbb{R} \text{ arbitrary } \textbf{values}$$

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Solution

$$f(x) = \sum_{i=1}^{n} y_i \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

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The conditions $f(1) = 1, f(2) = -1, f(3) = 2$ impose the following *linear* conditions in a, b, c :

$$\begin{aligned} a + \ \ b + c &= 1 & \qquad f(1) &= 1 \\ 4a + 2b + c &= -1 & \qquad f(2) &= -1 \\ 9a + 3b + c &= 2 & \qquad f(3) &= 2 \end{aligned}$$

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$$\begin{aligned} a + b + c &= 1 & \color{red}{f(1) = 1} \\ 4a + 2b + c &= -1 & \color{red}{f(2) = -1} \\ 9a + 3b + c &= 2 & \color{red}{f(3) = 2} \end{aligned}$$

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$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & -1 \\ 9 & 3 & 1 & 2 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & -19/2 \\ 0 & 0 & 1 & 8 \end{array}\right)$$

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Solution

The quadratic polynomial

$$f(x) = \frac{5}{2}x^2 - \frac{19}{2}x + 8 = \frac{1}{2}(5x^2 - 19x + 16)$$

does the trick.

System of linear equations

We are given distinct **points** $x_1, \ldots, x_n$ in $\mathbb{R}$ and arbitrary **values** $y_1, \ldots, y_n$ in $\mathbb{R}$ and want to find a polynomial $f(x)$ of degree $n-1$ such that for $i = 1, 2, \ldots, n$ we have

$$f(x_i) = y_i$$

Solution

There is a polynomial of degree $n-1$

$$f(x) = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$$

whose coefficients are given by the solution set of the linear system

$$\left(\begin{array}{cccccc|c} x_1^{n-1} & x_1^{n-2} & x_1^{n-3} & \cdots & x_1 & 1 & y_1 \\ x_2^{n-1} & x_2^{n-2} & x_2^{n-3} & \cdots & x_2 & 1 & y_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ x_n^{n-1} & x_n n-2 & x_n^{n-3} & \cdots & x_n & 1 & y_n \end{array}\right)$$

Extended Example 1

Can we find a polynomial $f(x)$ such that

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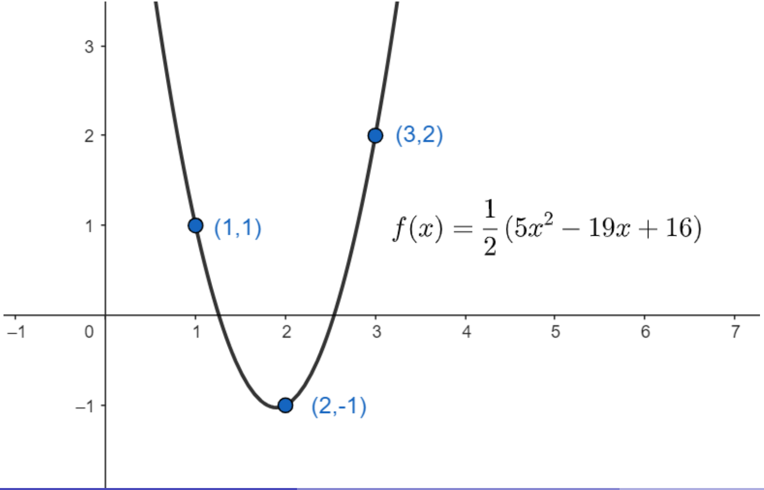

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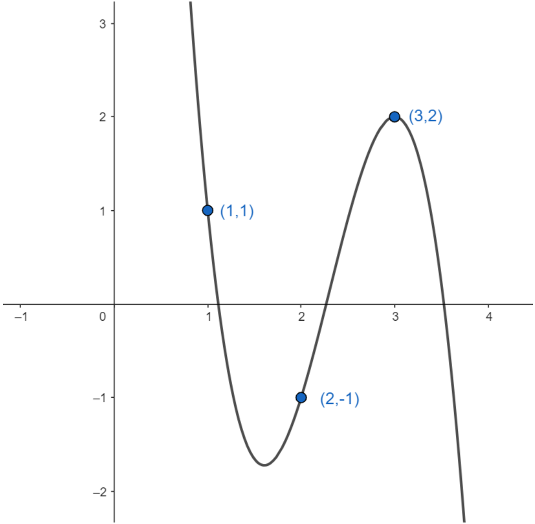
The cubic polynomial

$$f(x) = \frac{-11}{4}x^3 - 19x^2 - \frac{159}{4}x + \frac{49}{2}$$

does the trick.



Example 2

What degree polynomial would have the following conditions

$$f(1) = 1, \quad f(2) = 2, \quad f(3) = 3,$$
$$f'(1) = 4, \quad f'(2) = 5, \quad f'(3) = 6?$$

Example 2

What degree polynomial would have the following conditions

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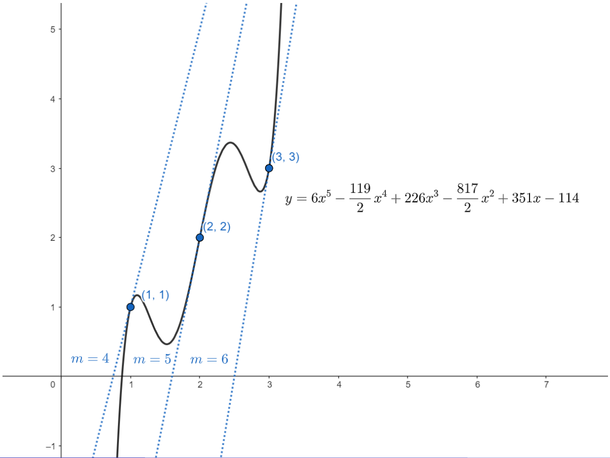


Observation

If there are n **conditions**

$$f^{(k_{i,j})}(x_{i,j}) = y_{i,j}$$

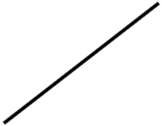
then there is a polynomial $f(x)$ of degree $n-1$ that simultaneously satisfies the n conditions (assuming n bounds $\{k_{i,j}\}$).
The **coefficients** of $f(x)$ are given a system of linear equations.

II. Multivariable polynomial interpolation

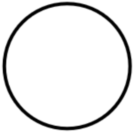
A **plane curve** is the set of points in $\mathbb{R}^2$ satisfying

$$f(x, y) = 0$$

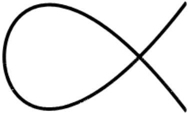
for some polynomial $f(x, y)$.



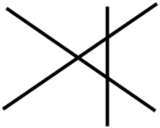
$x - y = 0$



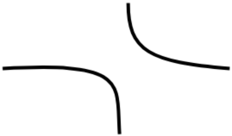
$x^2 + y^2 - 1 = 0$



$y^2 - x^3 - x^2 = 0$



$(y - x)(y + x)(x - 1) = 0$



$xy - 1 = 0$

$y^2 - x^3 = 0$

Given a point $p = (x, y)$ and a polynomial $f(x, y)$, then $f(p) = 0$ iff p lies on the plane curve $f(x, y) = 0$.

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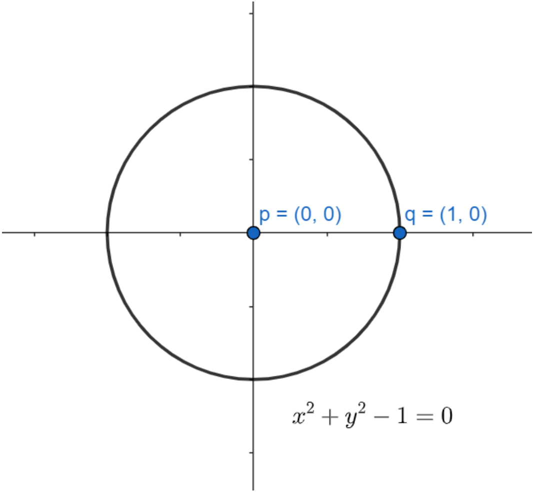

$f(q) = f(1, 0) = 1^2 + 0^2 - 1 = 0 \Rightarrow$ the point q lies on the circle
$f(p) = f(0, 0) = 0^2 + 0^2 - 1 \neq 0 \Rightarrow$ the point p **does not** lie on the circle

Example 3

Consider 3 points p_1, p_2, p_3 in the plane. What is the minimum degree d polynomial f such that $f(p_i) = 0$?

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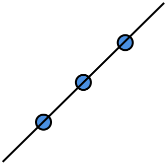
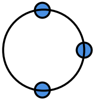
3 collinear points vs. 3 non-collinear points





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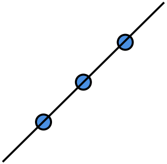

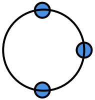

Problem

In the single-variable case, the position of the points **do not** matter.
In the multi-variable case, the position of the points **do** matter.

Example 4

Any five (general) points determine a unique conic

Let $F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$. Denote $p_i = (x_i, y_i)$ for $i = 1, 2, 3, 4, 5$.

Example 4

Any five (general) points determine a unique conic

Let $F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$. Denote $p_i = (x_i, y_i)$ for $i = 1, 2, 3, 4, 5$. Each point imposes a linear condition on a, b, c, d, e, f.

$$\begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Observe that this is a linear system of 5 equations and 6 variables, so we obtain a non-trivial solution and this solution uniquely determines $F(x, y) = 0$.

Example 4 cont.

Six (general) points do not lie on a conic

An additional point gives an **additional linear condition**. Therefore we have the linear system

$$\begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \\ x_6^2 & x_6y_6 & y_6^2 & x_6 & y_6 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

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Since the six points are random enough, the matrix is invertible. By the Invertible Matrix Theorem the only solution to this system is the trivial solution. Hence no conic passes through all six points.

Suppose polynomials of degree d have n coefficients.

- If we have fewer than n points, then there is always a plane curve of degree d passing through the points.
- If we have at least n points and the points are sufficiently general, then there is no plane curve of degree d passing through the points.

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- There is no conic curve passing through 6 general points.

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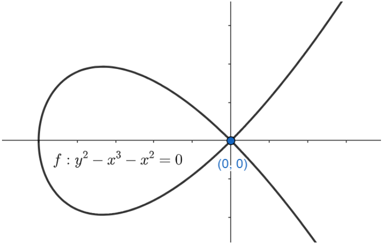
Examples

- There is a conic passing through any 5 points.
- There is no conic curve passing through 6 general points.
- There is a cubic curve passing through any 9 points.
- There is no cubic curve passing through 10 general points.

III. Plane curves with multiplicity

Example 4

Let $f(x, y) = y^2 - x^3 - x^2$ and $p = (0, 0)$.



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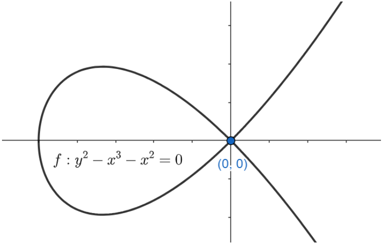

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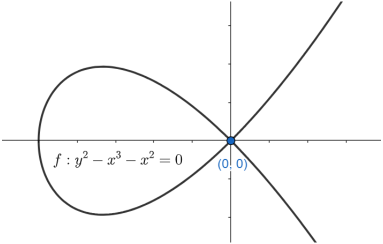

$$f(p) = 0^2 - 0^3 - 0^2 = 0$$

$$\frac{\partial f}{\partial x} = -3x^2 - 2x$$

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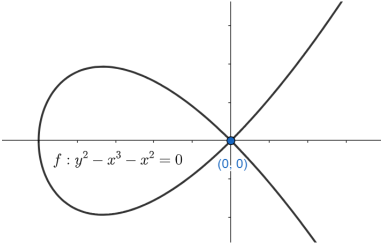

$$f(p) = 0^2 - 0^3 - 0^2 = 0$$

$$\frac{\partial f}{\partial x} = -3x^2 - 2x \Longrightarrow \frac{\partial f}{\partial x}(p) = -3(0)^2 - 2(0) = 0$$

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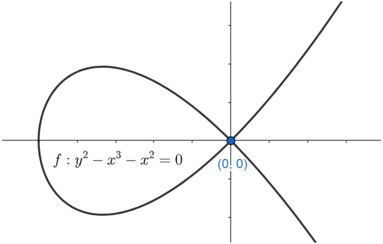

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$$\frac{\partial f}{\partial y} = 2y$$

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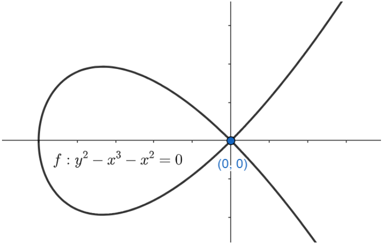

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$$\frac{\partial f}{\partial y} = 2y \Longrightarrow \frac{\partial f}{\partial y}(p) = 2(0) = 0$$

Conclusion

The cubic polynomial $f = y^3 - x^2 - x^2$ satisfies

$$f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0$$

at $p = (0, 0)$. We say that f **has multiplicity** 2 **at** p**.**

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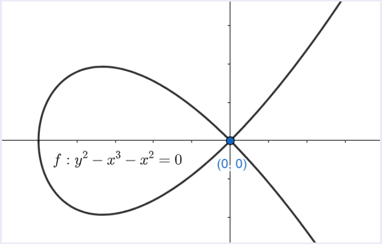
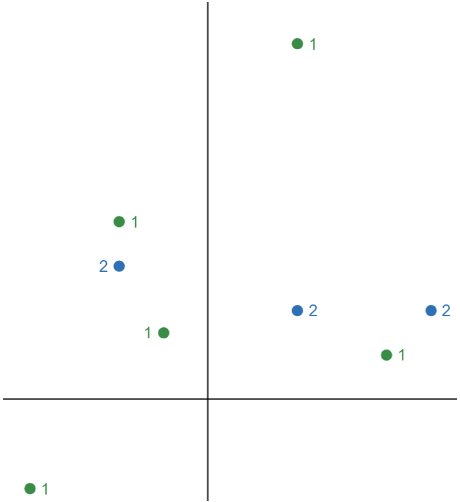
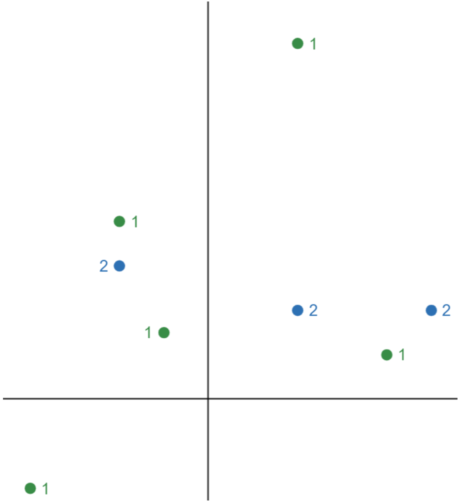
Geometrically, we can "see" that f has multiplicity 2 at p since the curve $f = 0$ passes through p twice. We say that p is a **double point**.



Example 5



Example 5



Each double point imposes **three** conditions ($f = \partial f/\partial x = \partial f/\partial y = 0$) and each simple points imposes **one** condition ($f = 0$).

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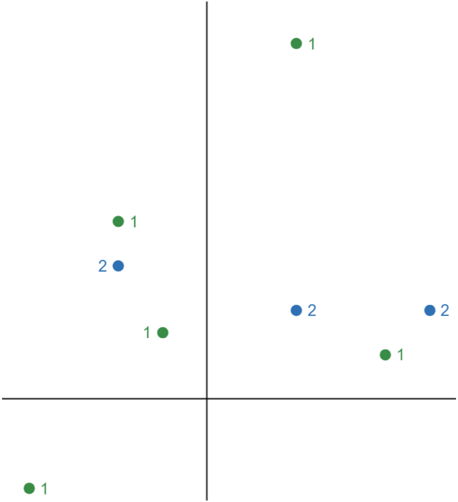

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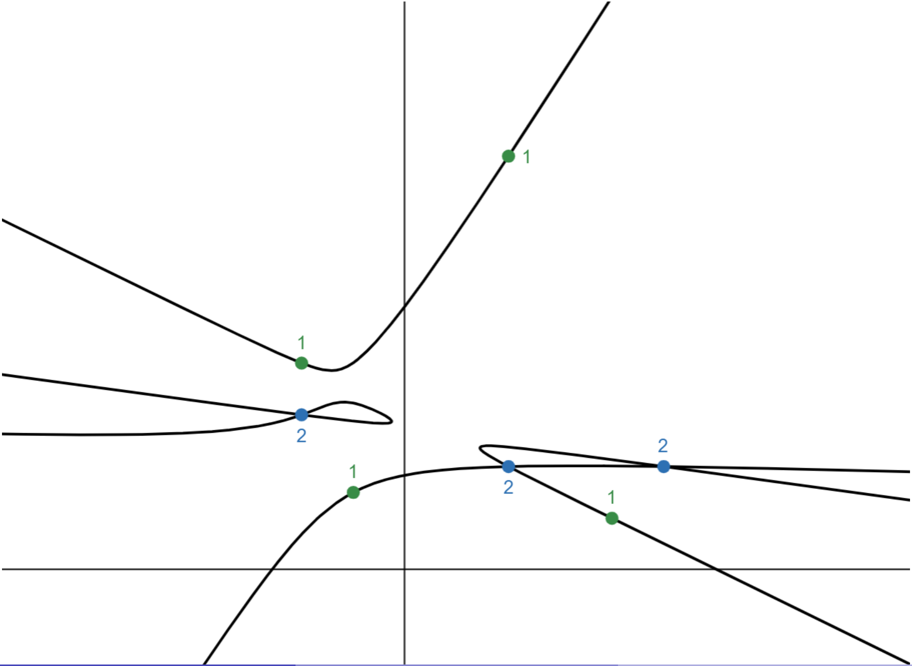
$$3 \cdot 3 + 5 \cdot 1 = 14 \text{ conditions}$$

Example 5 cont.

A quartic (degree 4) polynomial in two variables x, y has 15 coefficients:

$$x^4, x^3y, x^3, x^2y^2, x^2y, x^2, xy^3, xy^2, xy, x, y^4, y^3, y^2, y, 1.$$

Since there are more coefficients (15) than there are conditions (14), there is a quartic curve that has the prescribed conditions we want.


1
1
2
1
2
2
1
2
1

Example 6

Two double points p_1, p_2 impose **six**$(= 2 \cdot 3)$ conditions

$$F(p_1) = \frac{\partial F}{\partial x}(p_1) = \frac{\partial F}{\partial y}(p_1) = F(p_2) = \frac{\partial F}{\partial x}(p_2) = \frac{\partial F}{\partial y}(p_2) = 0.$$

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A quadratic polynomial in two variables x, y has **six** coefficients

$$F(x, y) = ax^2 + bxy + cy^2 + dx + ey + f\,.$$

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So there should be no solution...right?

Counterexample to Example 6

Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$. Let the line passing through the point p_1 and p_2 be given by $L(x, y) = 0$. But then

$$L(x, y)^2 = 0$$

is a conic curve that has multiplicity 2 at the points p_1 and p_2.

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Five double points impose **fifteen**$(= 5 \cdot 3)$ conditions.

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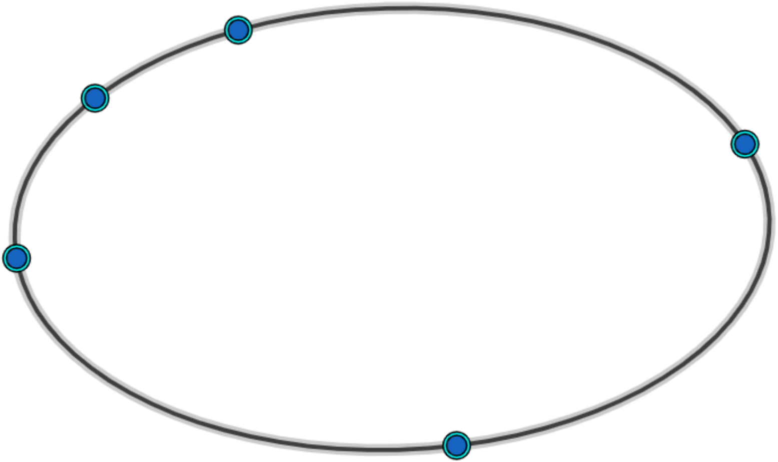

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SHGH-Conjecture

If the number of conditions a collection of points of arbitrary multiplicity equals the number of coefficients, then existance of a curve with these conditions relies on some number of copies of a curve stacked on top of each other.

IV. My research

Questions

- Given a set of points $p_1, \ldots, p_n$ in the plane with multiplicities $m_1, \ldots, m_n$, what is the minimal degree d polynomial f such that f has multiplicity at least m_i at p_i?

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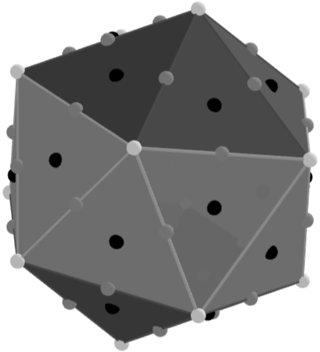
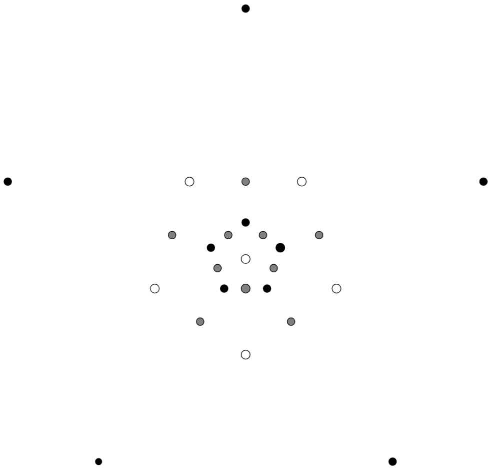
These are hard.

How we can alleviate the difficulty of this question?

- Consider **special** configurations of points.
- Exploit any symmetries/geometry of the configuration.

Geometry from reflection groups





Let $G = A_5 \times \mathbb{Z}_2$ be the symmetry group of the icosahedron.

The group G is generated by the following matrices.

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \frac{-1}{2}\begin{pmatrix} \omega - 1 & \omega & 1 \\ \omega & -1 & \omega - 1 \\ 1 & \omega - 1 & -\omega \end{pmatrix}.$$

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The group G acts naturally on the polynomial ring $S = \mathbb{C}[x, y, z]$.

Example

Let A be the matrix

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For example, A acts on the polynomial $x^2y - 2xy + 3z$ as:

$$A \cdot ({\color{red}x}^2 - 2{\color{red}x}{\color{blue}y} + 3z) = {\color{red}y}^2 - 2{\color{red}y}{\color{blue}z} - 3x$$

Question

What polynomials are **invariant** under the action of G? For what $f \in S$, does

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There are two other G-invariant polynomials of interest.

$$\phi_6 = x^4 y^2 + y^4 z^2 + x^2 z^4 + 4\,\omega\, x^2 y^2 z^2 - (\omega + 1)\,(x^2 y^4 + y^2 z^4 + x^4 z^2),$$

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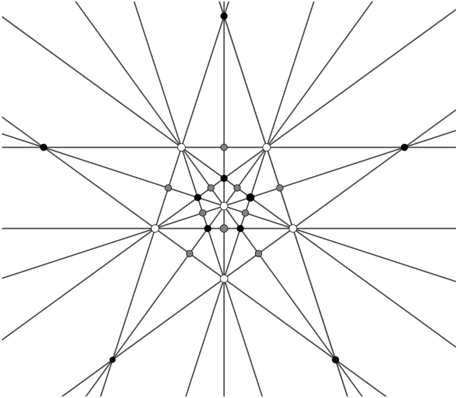
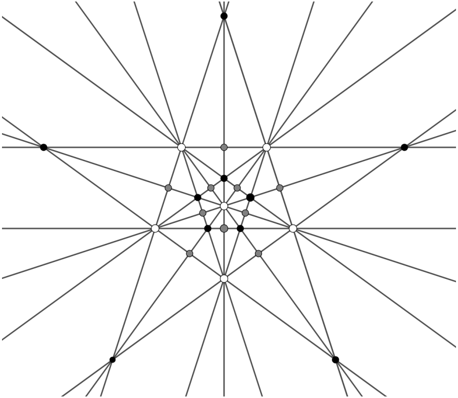
Example

$\phi_2^2\phi_6 - \phi_{10}$ is G-invariant.



There are double, triple and quintuple points of the line configuration.



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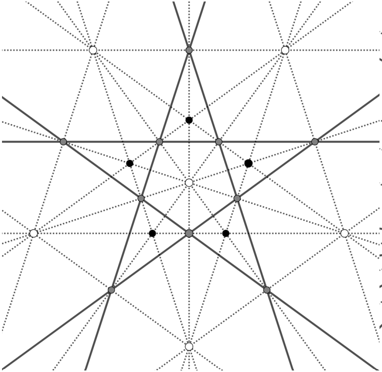
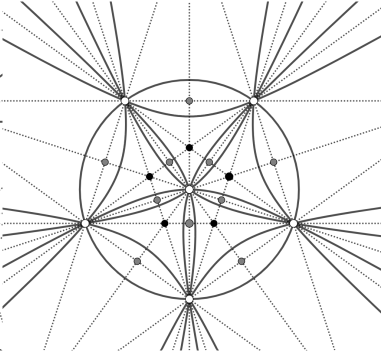
- There are 15 double points in the configuration.
- A double point imposes 3 linear conditions.
- These give 45 linear conditions to satisfy.
- Lemma states if one linear condition is satisfied, then the remaining 44 linear conditions are satisfied as well.

The polynomials $\phi_2, \phi_6, \phi_{10}$ are used to find curves that have multiplicities at some points of our initial configuration.



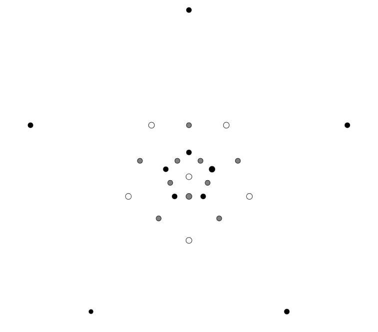
$(d, m) = (6, 2)$



$(d, m) = (12, 5)$

Initial question

What is the minimal ratio d/m of a polynomial of degree d with multiplicity m at the 31 points?



Calvo '22

The minimal ratio is $d/m = 11/2$.

Thank you!