n-POINTED CURVES AND MODULI SPACES

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1. Introduction

Section 2 is a brief revisit to the familiar geometry of $\mathbb{P}^1(\mathbb{C})$ with a new perspective to fit the language used in later sections. Sections 3 and 4 introduces ideas leading up to the definition of n-pointed curves. Sections 5 and 6 try to fix whats "wrong" with the definitions presented in 3 and 4. Lastly, we tie into what we have seen into the larger world of Moduli spaces.

2. Projective Complex Line

We denote $\mathbb{P}^1(\mathbb{C})$ to be the projective complex line, which can be thought of as the familiar Riemann Sphere. The automorphisms of $\mathbb{P}^1(\mathbb{C})$ form the group $\mathrm{PGL}_2(\mathbb{C})$, invertible 2×2 matrices over $\mathbb{C}$ with identification of two matrices given by one is a constant factor of the other.

Recall that $\mathrm{PGL}_2(\mathbb{C})$ acts 3-transitively on $\mathbb{P}^1(\mathbb{C})$. Let $p = (p_1, p_2, p_3, p_4)$ be an ordered set of points in $\mathbb{P}^1(\mathbb{C})$. Then there exists an automorphism ϕ_p which maps p_1 to 0, p_2 to 1 and p_3 to ∞. This map can be written as

$$\phi_p(z) = \frac{(z - p_1)(p_2 - p_3)}{(z - p_3)(p_2 - p_1)}$$

Let $\lambda(p)$ be the image of p_4 under ϕ_p. Observe that the first three points of p determines ϕ_p. We call a point $p = (p_1, p_2, p_3, p_4)$ a **quadruplet**. Quadruplets naturally live in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let Q be the set of all quadruplets that do not lie on the diagonals. In other words, this condition forces p_i to all be distinct. Then we can neatly describe the map $\lambda : Q \to \mathbb{P}^1(\mathbb{C})$ defined by $\lambda(p) = \phi_p(p_4)$. Since p_4 cannot be equal to p_1, p_2 or p_3, the image of λ is $\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}$.

We can now ramp things up. Two quadruplets $p, p' \in Q$ are **projectively equivalent** if there exists an automorphism $\phi \in \mathbb{P}^1(\mathbb{C})$ such that $\phi(p_i) = p'_i$. As we have mentioned before, $\mathrm{PGL}_2(\mathbb{C})$ acts only 3-transitively on $\mathbb{P}^1(\mathbb{C})$. Thus, not any pair of quadruplets are projectively equivalent. It stands to investigate how we can describe partition classes of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ we obtain from projective equivalence. An initial observation is that such quadruplets are projectively equivalent if and only if $\lambda(p) = \lambda(p')$.

We can reformulate $\lambda(p) = \lambda(p')$ as $p = (p_1, p_2, p_3, p_4)$ is projective equivalent to $p' = (0, 1, \infty, q)$ for a $q \in \mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}$. Since p_1, p_2 and p_3 determine ϕ as mentioned above, the mapping $\phi(p_4) = q$ was determined. So in fact, this value q is unique. In conclusion, we culminate these preliminary ideas to state the following

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proposition.

Proposition 2.1 The set of equivalence classes of Q is bijective to $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$.

3. Universal Families

A **family** is a variety B, paired with a projection map $\pi : B \times \mathbb{P}^1(\mathbb{C}) \to B$ and disjoint maps (also called **sections**) $\sigma_i : B \to B \times \mathbb{P}^1(\mathbb{C})$ for $i = 1, 2, 3, 4$, as displayed in the diagram below. The choice of 4 sections was selected to remain faithful to the context presented in section 2, although we will expand upon this choice later.

$$\begin{tikzcd} B \times \mathbb{P}^1(\mathbb{C}) \arrow[d, shift right=6, "\pi"'] \\ B \arrow[u, shift left=1] \arrow[u, shift left=3] \arrow[u, shift left=5] \arrow[u, shift left=7, "\sigma_i"'] \end{tikzcd}$$

As it turns out, Q is a variety. A family over Q can be given by the four sections $\sigma_i(p) = p \times p_i$. The importance of these sections is that $\sigma_i(p)$ for $p \in Q$ is precisely the i-th coordinate of p. Since every $p \in Q$ is projectively equivalent to $(0, 1, \infty, q)$, we have

$$\sigma_1(p) = Q \times 0$$

$$\sigma_2(p) = Q \times 1$$

$$\sigma_3(p) = Q \times \infty$$

$$\sigma_4(p) = Q \times q$$

We call these families **universal** because they carry the *universal property* via taking the *pull-back* of the family in question.

In Proposition 2.1, we mentioned these are the same as far as set theory is concerned...but we can do much more. Let us denote the equivalence classes of Q as $M_{0,4}$. The index 0 refers to the genus of $\mathbb{P}^1(\mathbb{C})$. The index 4 refers to the number of 'points in question', which in our case refers to our quadruplets. We will later define these points as *marks*. The set $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ is a variety in $\mathbb{P}^1(\mathbb{C})$. Imposing a universal family on $M_{0,4}$ will allow us to say that $M_{0,4}$ is in *natural bijection* with $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$. In this context, 'natural' refers to an adoption of a geometric structure to a collection of equivalence classes.

Two families $(B, \sigma_i)_{i=1}^4$ and $(B, \sigma_i')_{i=1}^4$ are **equivalent** if there exists an automorphism $\phi : B \times \mathbb{P}^1(\mathbb{C}) \to B \times \mathbb{P}^1(\mathbb{C})$ such that $\pi = \phi \circ \pi'$ and $\sigma_i' = \phi \circ \sigma_i$.

Proposition 3.1 Two families $\sigma : B \to Q$ and $\sigma' : B \to Q$ are equivalent if and only if $\lambda \circ \sigma = \lambda \circ \sigma'$.

Proof. Suppose $\lambda \circ \sigma = \lambda \circ \sigma'$ for any $p \in Q$. Then the quadruplets $\sigma(p)$ and $\sigma'(p)$ are projectively equivalent. That is, there exists an automorphism ϕ that takes $p_i \mapsto p_i'$ where $\sigma(p) = (p_1, p_2, p_3, p_4)$ to $\sigma(p') = (p_1', p_2', p_3', p_4')$. Thus these families are projectively equivalent. $\square$

4. n-POINTED CURVES

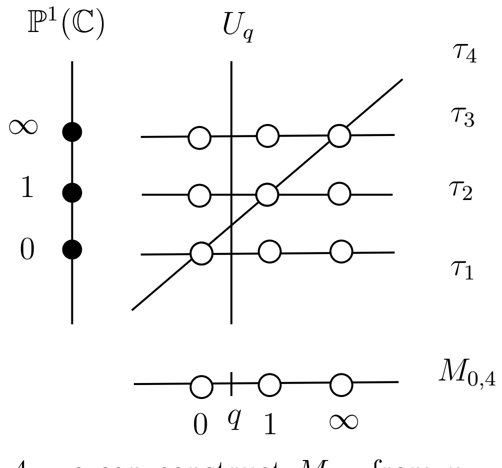
An n-**pointed smooth rational curve** $(C, p_1, ..., p_n)$ is a projective smooth rational curve C with $p_1, ..., p_n$ choice of n distinct points we call **marks**. An example of such a curve is $M_{0,4}$, where $n = 4$. An **isomorphism** of n-pointed rational curves C and C' is an isomorphism $\varphi : C \to C'$ which respects the order of marks, $\varphi(p_i) = (p_i')$ for $i = 1, ...n$.

A **family of** n-**pointed smooth rational curves** is a map $\pi : X \to B$ and n disjoint sections $\sigma_i : B \to X$ such that for any $b \in B$, the fiber of $\pi^{-1}(b)$ is a projective smooth rational curve, and for each $i = 1, ..., n$, $\sigma_i(b)$ gives the n marks of the fiber.

Example 4.1 Let $n = 3$. This reduces to using the fact that $\mathrm{PGL}_2(\mathbb{C})$ acts 3-transitively on $\mathbb{P}^1(\mathbb{C})$. That is, any (C, p_1, p_2, p_3) is isomorphic to $(\mathbb{P}^1(\mathbb{C}), 0, 1, \infty)$. Hence,there is only one isomorphism class and $M_{0,3}$ is a single point.

Example 4.2 Let $n = 4$. We previously discussed that any curve C with 4 distinct marks (C, p_1, p_2, p_3, p_4) is isomorphic to $(\mathbb{P}^1(\mathbb{C}), 0, 1, \infty, q)$ for $q \in \mathbb{P}^1(\mathbb{C})\backslash\{0, 1, \infty\}$. Just as with Q, we can impose a universal family on $M_{0,4}$ which we will call $U_{0,4}$

Just as we had three constant sections for the family over Q, we have three constant sections $\tau_1(q) = M_{0,4} \times 0, \tau_2(q) = M_{0,4} \times 1$ and $\tau_3(q) = M_{0,4} \times \{\infty\}$. Given whatever point $p \in M_{0,4}$, $\tau_i(q)$ will give us the i-th marked point of q, which is always either $0, 1$ or ∞ for $i = 1, 2, 3$. Let $\tau_4(q) = M_{0,4} \times q$. We refer to τ_4 as the 'diagonal' section. Since $q \neq 0, 1$ or ∞, we see that these sections are disjoint. The fiber of a point $q \in M_{0,4}$, denoted U_q, is a copy of $\mathbb{P}^1(\mathbb{C})$ with four points "marked", one point for each section.



Example 4.3 For $n \geq 4$, we can construct $M_{0,n}$ from $n - 3$ products of $M_{0,4}$ and removing the diagonals. As expected, its the first three sections of its universal family $U_{0,n}$ are the constant sections for $0, 1, \infty$. The remaining sections are induced in such a way that $\pi \circ \tau_i = \mathrm{id}|_{M_{0,n} \times \mathbb{P}^1(\mathbb{C})}$.

Since $\mathbb{P}^1(\mathbb{C})\backslash\{0, 1, \infty\}$ is not compact, we cannot expect $M_{0,n}$ to be compact. Namely, the property of compactness that goes wrong is the closedness of limits. The following example highlights this issue.

Example 4.4 Consider the two family of quadruplets

$$C_t = (0, 1, \infty, t) \; D_t = (0, t^{-1}, \infty, 1)$$

The cross ratio of C_t and D_t are

$$\lambda(C_t) = \frac{(-\infty)t}{t - \infty}, \qquad \lambda(D_t) = \frac{t^{-1} - \infty}{(1 - \infty)(t^{-1})}$$

Since the cross ratios are inverses of one another, then they obtain the same cross ratios and thus are isomorphic families. However as we let $t \to 0$, C_0 gives $p_1 = p_4 = 0$ while D_0 gives $p_2 = p3 = \infty$. Thus while projective equivalence allowed us to construct $M_{0,n}$, it simultaneously brings with it an issue that invalidates compactification of $M_{0,n}$ in the obvious way (that is, allow marked points to coincide). We take a brief pause on this discussion to introduce a tool that helps fix this issue.

5. Trees and Stability of n-pointed curves

A **tree of projective lines** is a connected curve composed of **twigs**, irreducible components of a tree, satisfying the following:

(1) Each twig is isomorphic to a projective line $\mathbb{P}^1(\mathbb{C})$.
(2) Each **node**, a point of intersection of twigs, is counted with multiplicity 2.
(3) There are no closed circuits.

For convenience, we will simply just say 'tree'. A point of C is said to be **special** if it is either a mark or node.

Let $n \geq 3$. An n-pointed rational curve C is said to be **stable** if C is a tree such that

(4) Each mark is a smooth point of C.
(5) Each twig has at least three special points.

These trees should reminisce of graphs (vertices and edges). We could have actually defined stability in terms of graph morphisms (incidence-preserving maps). An automorphism of $(C, p_1, ..., p_n)$ is an automorphism $\phi : C \to C$ that fixes each mark.

Proposition 5.2 A n-pointed curve C is stable if and only if there are no non-trivial automorphism of C.

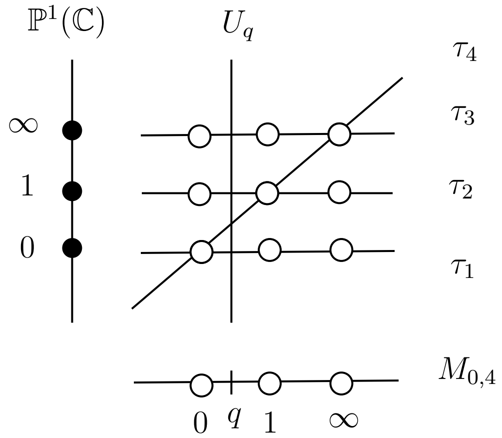
Proof. ($\Rightarrow$) Let C be a a stable n-pointed curve. Let ϕ be an automorphism of C. Since ϕ fixes each mark, then it must also map each marked twig to itself by definition. If ϕ fixes nodes, then we are done. We use an induction argument.

If a twig has 1 node, then it must have at least two marks by the stability of C. Since this node is an element of two twigs, then it must be that this node is fixed (since ϕ maps twigs onto itself). Thus ϕ fixes all special points of a twig. Since every twig has at least three of these, the automorphism must be trivial (since $\mathrm{PGL}_2(\mathbb{C})$ acts 3-transitively on $\mathbb{P}^1(\mathbb{C})$).

($\Leftarrow$) Let ϕ be the only (hence trivial) automorphism of C. There does not exist a twig of just one special point, or else there would be a non-trivial automorphism. Similarly for a twig with only two special points. So every twig must have at least three special points and C is stable. $\square$

6. Compactification of $M_{0,n}$

In Example 4.4, we saw that our definition of projective equivalence causes some issues for compactification. Restoring the three missing points $(0, 1, \infty)$ so that the compactification of $M_{0,4}$ is $\mathbb{P}^1(\mathbb{C})$ is not enough. In the diagram of Example 4.2 (also displayed below), restoring the missing points would also disrupt the disjointness of our sections. That is, if we were to consider U_q, we would get that $q = 0$ is marked twice (since τ_1 and τ_4 meet at that point).



To compactify $M_{0,4}$, we use the geometric operation on a variety known as a **blow-up**. Essentially, blowing up means replacing a particular point with a copy of $\mathbb{P}^1(\mathbb{C})$. We blow-up at points that disrupt the structure we are hoping to preserve. When we blow-up, its crucial to investigate what becomes of the universal family.

Thus restore the bad points into $M_{0,4}$ to obtain a copy of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, a variety we are comfortable working with. We blow-up $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ at the points $(0, 0)$, $(1, 1)$ and (∞, ∞). At $(0, 0)$, we insert a copy of $\mathbb{P}^1(\mathbb{C})$. Since we are only messing around with these three points, the fibers of π remain the same as before. That is, for $\pi^{-1}(q)$ is a copy of $\mathbb{P}^1(\mathbb{C})$ with $0, 1, \infty$ and q as marked points for $q \neq 0, 1, \infty$.

In particular, for $q = 0$, $\pi^{-1}(0)$ is two copies of $\mathbb{P}^1(\mathbb{C})$ intersecting at a single point. The new copy of $\mathbb{P}^1(\mathbb{C})$ obtained from blowing up is called the **exceptional divisor** E_0 of the point 0. Denote $\pi^{-1}(0)$ by the union of U_0 and E_0. We want to identify the marked points on this fiber. Since 1 and ∞ are far enough away from the blow-up, 1 and ∞ are marked points on U_0. In other words, the sections τ_2 and τ_3 remain disjoint on U_0 after blowing up. But they do not intersect E_0 because of the uniqueness of the intersection of E_0 and U_0. In fact, τ_1 and τ_4 do intersect E_0 but now they are disjoint in the blow-up. Also including the point of intersection, we have that U_0 and E_0 are both twigs since they have three special points! Similar scenarios occur for the fibers at $q = 1, \infty$. We denote the compactification of $M_{0,4}$ as $\overline{M}_{0,4}$.

The idea of how blow-ups solve our compactness problem is that as two distinct marks tend towards each other, a new $\mathbb{P}^1(\mathbb{C})$ is formed between them. This new $\mathbb{P}^1(\mathbb{C})$ is in fact a twig, since it carries the two distinct marks and the point of intersection of original $\mathbb{P}^1(\mathbb{C})$ that contained the marks.

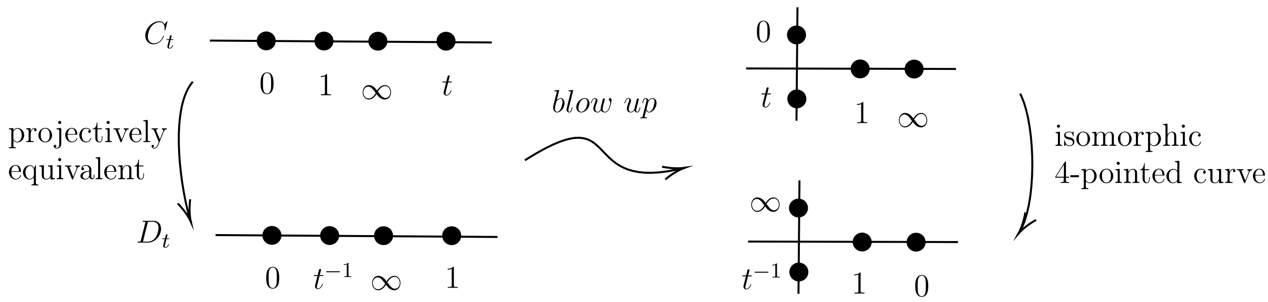
Example 6.1 We continue Example 4.4. We had the following set up: Consider families

$$C_t = (0, 1, \infty, t), \quad D_t = (0, t^{-1}, \infty, 1)$$

The cross ratios of C_t and D_t are

$$\lambda(C_t) = \frac{(-\infty)t}{t - \infty}, \qquad \lambda(D_t) = \frac{t^{-1} - \infty}{(1 - \infty)(t^{-1})}$$

We saw that they limit towards different values as $t \to 0$ despite the families being projectively equivalent. We employ the diagrams of corresponding trees, for a given t value, to make the punchline pictorial.



As t tends toward 0, p_4 converges to p_1 in C_t. In D_t, p_2 converges to p_3. Blowing up $M_{0,4}$ makes it so that the convergence is halted once the exceptional divisor comes into play. Then, p_4 and p_1 remain distinct (p_2 and p_3 respectively) and so the exceptional divisor becomes a twig of the curve. Since both twigs have at least three special points, these are stable. In fact, C_t and D_t in the blow-up are isomorphic as stable 4-pointed curves.

From the last paragraph, it gives us some intuition behind the following remark by Kock: "In particular, the points of the variety of $\overline{M}_{0,n}$ are in bijective correspondence with the set of isomorphism classes of stable $n-$pointed rational curves.

For a given t_0, C_{t_0} and D_{t_0} are projectively equivalent. Thus we *want* to consider them the "same" in $M_{0,4}$, but as we have hammered down enough, issues arise. The want becomes a reality when we compactify and obtain $\overline{M}_{0,4}$. The curves C_{t_0} and D_{t_0} become isomorphic 4-pointed curves. Just as in $M_{0,4}$ we consider isomorphism classes of quadruplets, we consider isomorphism classes of 4-pointed curves in $\overline{M}_{0,4}$.

7. Moduli Spaces

What initially began as a project on Moduli Space had to be renamed to be "n-pointed curves and Moduli Spaces". This begs the question, what does $n-$pointed rational curves have to do with Moduli Spaces. Or rather, what is a Moduli space and what do n-pointed rational curves have to do with them?

A **moduli problem** consists

∘ a class of objects P in a category $\mathcal{C}$

∘ a *family* of objects over a base variety B and a notion of an equivalence of families

A **Moduli Space** is a space M that classifies the equivalence classes of the objects $\mathcal{P}$ in the same category $\mathcal{C}$. More precisely,

∘ elements of M are in bijection with the equivalence classes of objects $\mathcal{P}$ in $\mathcal{C}$

∘ For any family over a variety B, we obtain a map $B \to M$

In essence, a Moduli space parameterizes equivalance classes. It is a tool that allows us to see how a space "moves" in regards to a particular property (or class). This paper focused on the very neat and simple case of where the objects were n-tuples of points on $\mathbb{P}^1(\mathbb{C})$, the equivalence of such objects being projective equivalence and our families being the sections σ_i.

The incredible aspect of this construction is that it dubbed the term 'space' in its name. It inherits a geometric structure from an algebraic variety. From this, we are able to talk about the compactification of the space and inquire what does compactification do to the equivalence relation and parameters that laid the groundwork for the construction of the space. It tinkers some structure but we were able to patch it up.

In particular, we only considered spaces of genus 0. The following theorem by Finn Knudsen gives us that we are able to construct compactifications of $M_{0,n}$:
Theorem 7.1 For each $n \geq 3$, there is a smooth projective variety $\overline{M}_{0,n}$ which is a fine moduli space for stable n-pointed rational curves and contains $M_{0,n}$ as a dense open subset.

Moving forward, the marks I want to hit regarding Moduli Spaces are:

∘ Verify the smoothness of $\overline{M}_{0,n}$

∘ Learn what changes in genus 1

∘ Be able to identify the schemes floating around when we talk about Moduli spaces

8. References

J. Kock, I. Vainsencher. *Kontsevichâs Formula for Rational Plane Curves*. 1999.

R. Cavalieri. *Moduli Spaces of Pointed Rational Curves* (Lecture notes presented at Fields Institute). 2016.

Conversations with Steffen Marcus and Daniel Levine.