COVERING SPACES

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1. Introduction

In this paper, we discuss the theory of covering spaces. Before we give a formal presentation of it, we introduce an underlying tool to allow us to better understand covering spaces. We call such tool the *fundamental group*. Although this tool may at first glance not be useful to understand covering spaces, the main theorem, **Theorem 5.1**, relates these two concepts together.

In Section 2, we present a necessary background of topology, most of which is just a generalization of concepts used in analysis and algebra. In Section 3, we introduce the definition of the fundamental group of a topological space along with examples of these groups. In Section 4, we finally present covering spaces. We give some examples before we provide some lemmas about covering spaces. The examples coupled with these lemmas will prepare us for the fundamental theorem about covering spaces. Afterwords, we showcase some applications of this fundamental theorem. The author referenced [Hat02] and [Mas77] for this expository paper.

2. Background

A topological space is simply a set endowed with a structure that determines how subsets interact with one another. In analysis, the set we are primarily concerned with is $\mathbb{R}^n$. The subsets of $\mathbb{R}^n$ we like to think about are open intervals. Then when we bring in the union and intersection operations into the mix, we are familiar with the fact that the union and intersection of two open intervals is itself an open interval in $\mathbb{R}^n$. These open intervals are open sets in $\mathbb{R}^n$ When we make these statements rigorous, we have that $\mathbb{R}^n$ is a topological space, as we define below.

Let X be a set and τ be a collection of subsets of X. We say (X, τ) is a **topological space** if the following conditions are satisfied:

(1) The sets $\emptyset$ and X are in τ
(2) An arbitrary union of open sets is an open set
(3) A finite intersection of open sets is an open set

An element of τ is called an **open set** if it satisfies conditions (1)-(3). When the topology is clear from the context, we commonly refer to our space simply as X rather than as (X, τ).

We would like a construction that allows us to say when two topological spaces are the 'same'. A map of topological spaces $f : X \to Y$ is a **homeomorphism** if the following are satisfied:
(1) f is bijective
(2) f is continuous

(3) f^{-1} is continuous

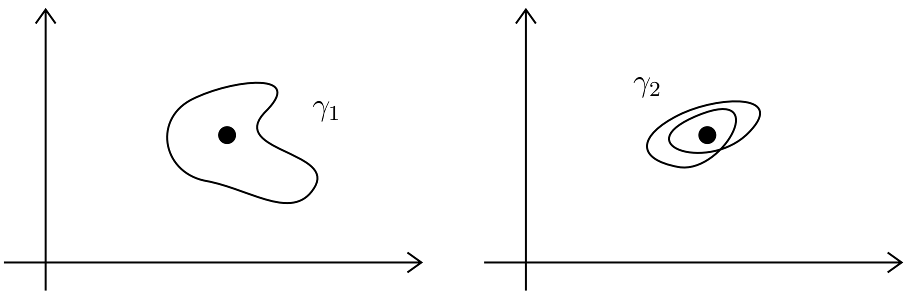
3. Fundamental Group

The fundamental group of a topological space gives a group structure to the classes of "loops" based at a point that exist in a space.
First we build up the language of fundamental groups, which begins with paths. We say a **path** in a space X is a continuous map $\gamma : I \to X$. Supposing X is $\mathbb{R}^n$, paths can be thought of as curves.

If we imagine our path as a piece of yarn sitting on flat surface, we can 'wiggle' the yarn so that it has a distinct shape from when we began. We want to think of two paths as the same if we can push one to another in this way. Formally, paths $\gamma_0, \gamma_1 : I \to X$ are **homotopic** if there is a continuous function $h : I \times I \to X$ such that $h(0,t) = \gamma_1(t)$ and $h(1,t) = \gamma_2(t)$. To denote homotopic paths we write $\gamma_1 \sim \gamma_2$. We use $\sim$ because homotopy is actually an equivalence relation. When needed, we will write the equivalence class of a paths as $[\gamma]$, the collection of paths that are homotopic to γ.

A **loop** is a path γ such that $\gamma(0) = \gamma(1) = x$. Call x the **basepoint** of γ. Let's consider the set of loops based at a basepoint x_0. Some of these loops are homotopic to one another. Thus we will consider them to be the same. Denote this set of homotopic classes by $\pi_1(X, x_0)$, where X is the space and x_0 is a the basepoint. We define an operation on the set by concatenation of loops based at x_0. That is, if f and g are loops with basepoints x_0, then fg is the path that travels along f in $0 \leq t \leq \frac{1}{2}$ and then travels along g in $\frac{1}{2} \leq t \leq 1$. Since $f(1) = g(0)$, the concatenation makes sense. Associativity comes naturally as well since the loops preserve basepoints. If $\gamma(t)$ is a loop, then $\gamma^{-1}(t) := \gamma(1-t)$. We see that $[\gamma][\gamma^{-1}]$ is actually just the constant x_0 path. We also see that the constant basepoint path $e(t) = x_0$ is the identity. So π_1 is in fact a group, called the **fundamental group**. Let's check out some examples.

Example 3.1 Let X be $\mathbb{R}^2$ with an arbitrary point y removed. In the following figure, let γ_1 go around the point once counterclockwise and γ_2 go around the point twice counterclockwise. We let counterclockwise orientation be positive. These are in fact not homotopic since we cannot push γ_1 to γ_2.

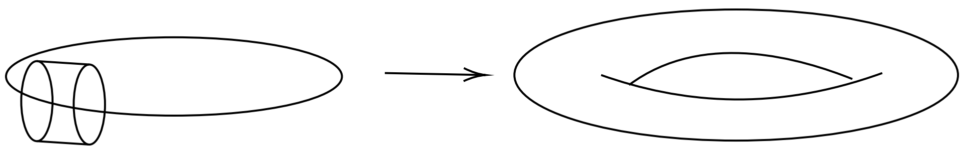


The loop γ_1 can be associated with going around the point once in positive orientation, γ_2 can be associated for going around the point twice in the positive orientation. If a γ does not go around the missing point, then γ is homotopic to the constant path.

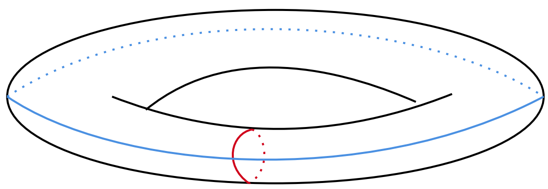
Otherwise, γ can be associated to an integer, positive or negative, according to how many times it goes around the point in the positive or negative direction. Thus $\pi_1(\mathbb{R}^2 - y) = \mathbb{Z}$.

Example 3.2 Consider X to be S^1, the unit circle. A well known theorem states that $\pi_1(S^1) = \mathbb{Z}$. We can consider the point $(1, 0)$ to be the basepoint if we were to embed S^1 in $\mathbb{R}^2$ in the usual fashion. Then using a similar argument as Example 3.1, we can consider the homotopy classes of S^1 to be represented by loops based at $(1, 0)$ that go around the circle n times, either in the positive or negative direction.

Example 3.3 We end this section by presenting the fundamental group of a torus. First we explain what a torus is. Suppose we have a horizontal copy of S^1 and then for each point in S^1, we put a vertical copy of S^1, as shown below. What we are essentially doing is creating tube around the horizontal copy of S^1. Once we place an S^1 at each point x in the horizontal copy of S^1, we get what resembles the surface of a donut, shown below.



In Example 3.2, we saw that $\pi_1(S^1) = \mathbb{Z}$. The torus, T^2, was built by placing copies of S^1 on S^1. Because of this, we can think of T^2 as the space $S^1 \times S^1$, since we can associate each point on T^2 as an ordered pair of (a, b) where a is some point on the first S^1 and b is some point on the second S^1. A neat fact about spaces that are built this way is that the fundamental group of T^2 is simply the product of the fundamental groups of S^1 and S^1. So $\pi_1(T^2) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$. We can see the blue S^1 act as the 'x-axis' and the red S^1 act as the 'y-axis' as seen below. Any loop can be thought of as how many times it wraps along the blue and red S^1 and in what direction.



The torus T^2

4. Covering Spaces

We finally arrive to the nexus of what this paper is about. Before we discuss what a covering space is, let's present a non-math analogy to motivate the definition of a covering space. Let's imagine the architecture of an apartment complex. Suppose an apartment complex contains many apartments that are the same and you live in one particular apartment. Consider your bedroom. Since the apartment complex contains many apartments that are identical, there are many bedrooms that look exactly like yours. Since the interiors of apartments are disjoint from one another,

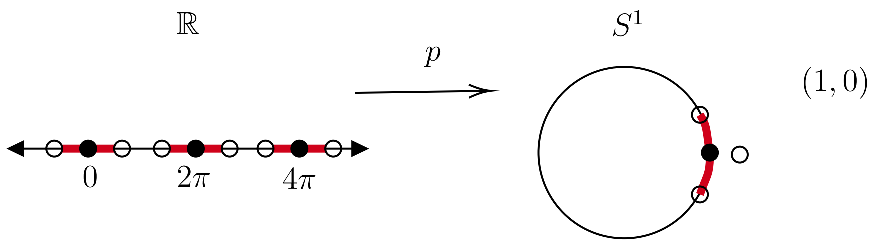


FIGURE 1. A covering space for S^1

surely we see that these bedrooms are disjoint from one another. We now build a mathematical notion which generalizes this idea.

Formally, a **covering space** of a space X is a pair $(\tilde{X}, p)$ consisting of a topological space $\tilde{X}$ and a continuous map $p : \tilde{X} \to X$ satisfying that each point $x \in X$ has an open neighborhood U in X such that $p^{-1}(U)$ is a union of disjoint open sets in $\tilde{X}$ where each open set is mapped homeomorphically onto U via p. Let's check out some examples.

Example 4.1 Let $p : \mathbb{R} \to S^1$ be defined by wrapping the line around the circle infinitely often so that that every multiple of 2π is the basepoint $(1, 0)$. Then we can think of $\mathbb{R}$ as the union of intervals $[2k\pi, 2(k+1)\pi]$ for $k \in \mathbb{Z}$. So every 2π period, we are travelling the circle once. As we consider a small neighborhood U around a point x, we get that $p^{-1}(U)$ to be the union of disjoint open intervals around the points in $\mathbb{R}$ that map to x. See Figure 1

Example 4.2 Let's consider the space $X = S^1$. Let $p : S^1 \to S^1$ be a covering map that sends (r, θ) to $(r, 2\theta)$. The map p wraps the circle around itself twice. Thus going around once in the range S^1 means we travel a semicircle in the domain S^1. For a neighborhood U of a point $x \in S^1$, $p^{-1}(U)$ will have two disjoint neighborhoods around the inverse images of x in each semicircle of S^1.
Before we dive into our next example, we showcase another way to construct the torus, this time using $I \times I$. Suppose we have that the edges of $I \times I$ are oriented such that the horizontal edges are facing same direction, as well as the vertical edges. See Figure 2 for reference.

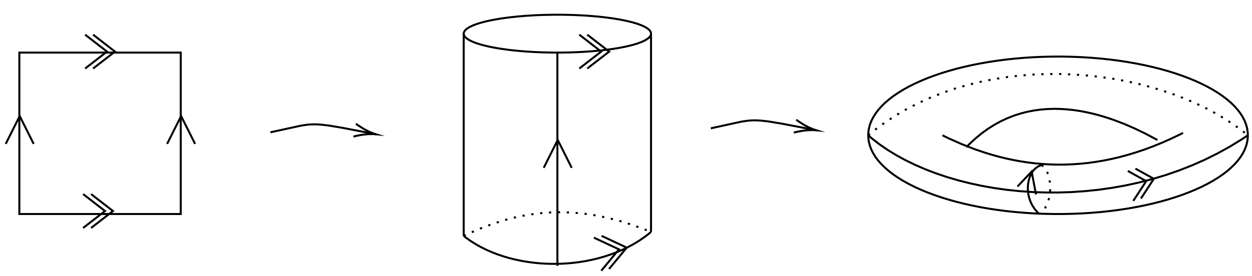


FIGURE 2. Gluing of $I \times I$ to get T^2

Then we can glue the vertical sides together such that the arrows meet in orientation-preserving directions. The resulting object we get is the open cylinder. It does not

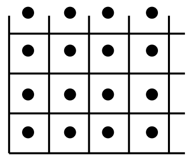


FIGURE 3. Similar points in the first quadrant under the equivalence relation

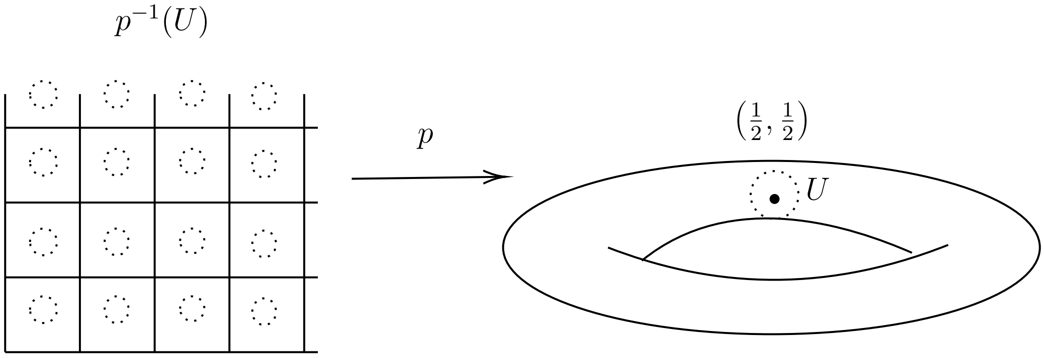


FIGURE 4. Inverse Image of a neighborhood of $(\frac{1}{2}, \frac{1}{2})$ in first quadrant

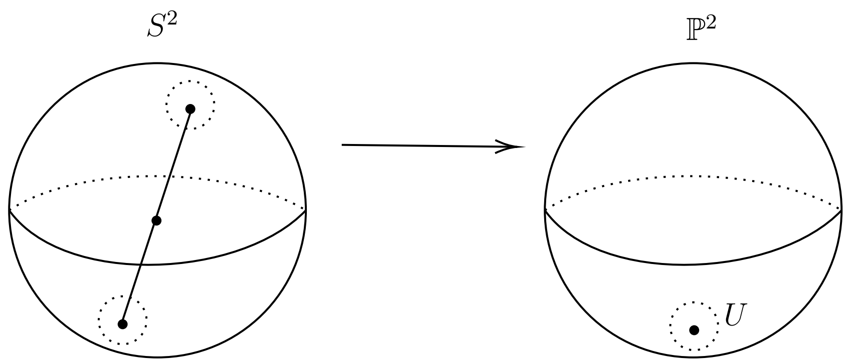
have a top or bottom lid. Then we perform a similar action to glue the top and bottom together such that the arrows agree in orientation. The resulting object we obtain is the torus.

One observation is that we can tile $\mathbb{R}^2$ with I^2's. We now introduce a way to relate these squares on $\mathbb{R}^2$ by an equivalence relation. For a point $(a, b) \in \mathbb{R}^2$, $(a, b) \sim (c, d)$ if and only if $a - c, b - d \in \mathbb{Z}$. For example, the point $(\frac{1}{2}, \frac{1}{2})$ is related to the points $(\frac{1}{2}, \frac{3}{2}), (\frac{-1}{2}, \frac{-1}{2})$ and $(\frac{7}{2}, \frac{35}{2})$. See Figure 3 for reference.

Example 4.3 Let $p : \mathbb{R}^2 \to T^2$. That is, T^2 can be thought of as $\mathbb{R}^2$ under the equivalence relation: $T^2 \cong \mathbb{R}^2/\sim$. Then it follows that if for any $(x_0, y_0) \sim (x_1, y_1)$, then $p((x_0, y_0)) = p((x_1, y_1))$. For example, consider an ϵ ball around $(\frac{1}{2}, \frac{1}{2}) \in T^2$. Then $p^{-1}(U)$ is a union of small neighborhoods around every $(\frac{k}{2}, \frac{l}{2})$ for $k, l \in 2\mathbb{Z} + 1$. See Figure 4.

For the next example, we will showcase a covering space for the projective plane $\mathbb{P}^2$. There are many ways to think of the projective plane. The way we will define $\mathbb{P}^2$ is S^2 with the relation that antipodal points are identified with one another, that is, glued with one another. So really we have that $\mathbb{P}^2 \cong S^2/\sim$ where $\sim$ relates antipodal points. On S^2 we will denote the antipodal point of a point a as $\overline{a}$. Clearly, the antipodal point of $\overline{a}$ is just a. This should be enough to understand the following example.

Example 4.4 Let $p : S^2 \to \mathbb{P}^2$. Let p be defined by taking a point on S^2 and mapping it to the set $\{a, \overline{a}\}$ on $\mathbb{P}^2$. Then consider a neighborhood U of $a \in \mathbb{P}^2$. Then $p^{-1}(U)$ is a disjoint union of neighborhoods around the points a and $\overline{a}$ in S^2.



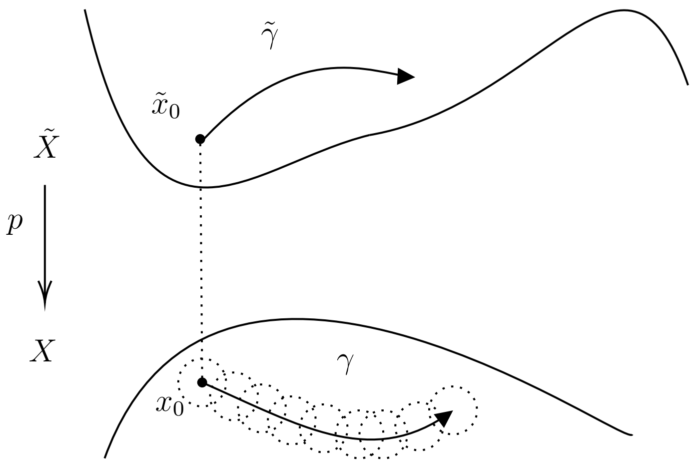
Next we introduce some ideas regarding covering spaces. A **lift** of a map $f : Y \to X$ of topological spaces is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$.

Example 4.5 Consider $f : [0, 1] \to S^1$ defined by $f(t) = (r, 2t\pi)$ and the covering map from Example 4.1. A lift $\tilde{f}$ of f is the map $\tilde{f} : [0, 1] \to \mathbb{R}$ defined by $\tilde{f}(t) = \pi t$. We have that f is the path on S^1 from 0 to π, and $\tilde{f}$ is the path on $\mathbb{R}$ from 0 to π. We see here somewhat trivially that $\tilde{f}$ is a lift of f.

With lifts at our disposal, we get an interesting property that allows us to relate paths in X and paths in $\tilde{X}$.

Lemma 4.6 Given $p : \tilde{X} \to X$ with $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ such that $p(\tilde{x}_0) = x_0$ and a path γ in X starting at x_0 then there is a unique lift $\tilde{\gamma}$ in $\tilde{X}$ starting at $\tilde{x}_0$ such that $p\tilde{\gamma} = \gamma$.

From Examples 4.1, 4.2 and 4.3, it is typically the case that for a point $x_0 \in X$, the cardinality of $p^{-1}(x_0)$ is greater than one. Lemma 4.6 begins with saying suppose we have a particular point $\tilde{x_0} \in p^{-1}(x_0)$. By the definition of a covering map, there is a neighborhood U of x_0 which is homeomorphic to each disjoint neighborhood in $p^{-1}(U)$. Since these open sets are disjoint, there is in fact only one particular open set in $p^{-1}(U)$ that contains $\tilde{x}_0$, call it $\tilde{U}$. This open set is homeomorphic to U, i.e., these are identical copies of one another. Since U contains a piece of a path γ, then $\tilde{U}$ will contain a piece of a corresponding path.



Proof. Let U_0 be a neighborhood of $\gamma(0) = x_0$ in X such that $p^{-1}(U_0)$ is a disjoint union of homeomorphic sets. Since p is a covering map, $p^{-1}(U_0)$ contains a single neighborhood $\tilde{U}_0$ containing $\tilde{x}_0$ such that U_0 and $\tilde{U}_0$ are homeomorphic. So there must exist a t_1 with $0 < t_1 < 1$ such that there is path γ_0 in U_0 with $\gamma_0(s) = \gamma(\frac{t}{t_1})$. Then there must be a path $\tilde{\gamma}_0 \subset \tilde{U}_0$ such that $p \circ \tilde{\gamma}_0 = \gamma_0$. Let U_1 be a neighborhood of $\gamma(t_1) = x_1$ such that $p^{-1}(U_1)$ is a disjoint union of homeomorphic sets. Let $\tilde{U}_1 \subset p^{-1}(U_1)$ be the neighborhood with nonempty intersection with $\tilde{U}_0$. So there must a exist a t_2 such that $t_1 < t_2 < 1$ and there is a path $\gamma_1 \subseteq U_1$ with $\gamma_1(0) = x_1$, $\gamma_1(1) = \gamma(t_2)$ and $\gamma_1(s) = \gamma(\frac{t-t_1}{t_2-t_1})$. Again, there must be a path $\tilde{\gamma}_1 \subset \tilde{U}_1$ such that $p \circ \tilde{\gamma}_1 = \gamma_1$ since $\tilde{U}_1$ and U_1 are homeomorphic. We can continue this process, obtaining triples (x_k, U_k, γ_k) where U_k is a neighborhood of $x_k = \gamma(t_k)$ for some t_k such that $p^{-1}(U_k)$ is a disjoint union of homeomorphic sets and the path $\gamma_k(0)$ is given by $\gamma_k(0) = x_k$, $\gamma_k(1) = \gamma(t_k)$ and $\gamma_k(s) = \gamma(\frac{t-t_k}{t_{k+1}-t_k})$ for some t_{k+1}. Then we obtain a path $\tilde{\gamma}_k$ such that $p \circ \tilde{\gamma}_k = \gamma_k$. We can choose finitely many neighborhoods U_i covering γ since γ is a compact subset. So in fact, we will have $l+1$ neighborhoods covering γ and $t_l = 1$ for some $l \in \mathbb{N}$. Thus we let $\tilde{\gamma} = \tilde{\gamma}_l \circ \tilde{\gamma}_{l-1} \circ ... \circ \tilde{\gamma}_0$ and $\tilde{\gamma}$ is the unique lift of γ starting at $\tilde{x}_0$. □

Remark Notice that in the proof of Lemma 4.6, we state twice '$\tilde{\gamma}_0 \subset \tilde{U}_0$'. Technically, the path $\tilde{\gamma}_0$ is a function and therefore cannot be a subset of a neighborhood $\tilde{U}_0$. What we mean is the *image* of $\tilde{\gamma}_0$ is a subset of $\tilde{U}_0$. Of course, we intuitively think of $\tilde{\gamma}_0$ as a curve but precision with language must be preserved.

In example 4.5 and Lemma 4.6, we have considered lifts of functions which were paths. Next, we informally give the idea of how to lift homotopies. Given two homotopic paths $\alpha, \beta : I \to X$, there exists a continuous function $h : I \times I \to X$ that relates these two paths, refer back to Section 3 for reference. We think of h as the collection of paths we obtain from moving α to β. For example, $h(.5, t)$ is a path that is the 'halfway' path between α and β. Supposing X has a covering space $\tilde{X}$, then there is a $\tilde{h}(.5, t)$ that is a lift of $h(.5, t)$. However $s = .5$ was arbitrary. For any $s \in [0, 1]$, we can lift $h(s, t)$ to obtain a family of lifts $\tilde{h}(s, t)$. The upshot is to recognize that $\tilde{h}(0, t) = \tilde{\alpha}$ and $\tilde{h}(1, t) = \tilde{\beta}$. So by lifting every path $h(s, t)$, we in fact obtain a family of paths that relate $\tilde{\alpha}$ and $\tilde{\beta}$. This gives the following lemma, which we will end the section with.

Lemma 4.7 If $p : \tilde{X} \to X$ is a covering space and $\alpha \sim \beta$ in X with starting point x_0, then there exists homotopic paths $\tilde{\alpha}$ and $\tilde{\beta}$ in $\tilde{X}$ starting at some point $\tilde{x}_0 \in p^{-1}(x_0)$.

5. The Induced Monomorphism

We now present the main theorem. This theorem elegantly relates the fundamental groups of a base space X and a covering space $\tilde{X}$.

Theorem 5.1 The associated map $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ of $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$, defined by $p_*(\gamma) = p \circ \gamma$ for $\gamma \in \pi_1(\tilde{X}, \tilde{x}_0)$, is an injective group homomorphism.

Proof:To show p_* is a homomorphism, we just can convince ourselves that going around loops in $\tilde{X}$ is the same as going around loops in X. We were using this fact that $\tilde{\gamma}$ and γ were the 'same' but in different spaces. So p_* is indeed a homomorphism. Let $\tilde{\gamma}$ be a loop based at $\tilde{x}_0$ in $\tilde{X}$ such that $p \circ \tilde{\gamma}$ is the trivial loop e in $\pi_1(X, x_0)$. Also notice that $p \circ \tilde{e}$, the trivial loop in $\pi_1(\tilde{X}, \tilde{x}_0)$, is also e. So we have that $(p \circ \tilde{\gamma}) \sim (p \circ \tilde{e})$. By Lemma 4.7, we have that $\tilde{\gamma} \sim \tilde{e}$. This gives that $\tilde{\gamma}$ is in the same homotopy class as $\tilde{e}$. So $\text{Ker}(p_*) = \{[e]\}$ thus p_* is injective. $\square$

By a similar argument, any continuous map $f : X \to Y$ of topological spaces with $f(x_0) = y_0$ for $x_0 \in X$ and $y_0 \in Y$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ of associated fundamental groups of each topological space. What makes a covering map special is that its induced homomorphism of fundamental groups is an injective. A consequence of Theorem 5.1 is that the image of p_* is a subgroup in $\pi_1(X, x_0)$. We now revisit the examples in Section 4 and see what the subgroups of $\pi_1(X, x_0)$ described by the image of p_* are.

Example 5.2 Let p be the covering map in Example 4.1. We have that $\pi_1(\mathbb{R})$ is the trivial group, since any loop is homotopic to a constant path. As mentioned in Example 3.2, $\pi_1(S^1) = \mathbb{Z}$. The group $\pi_1(\mathbb{R})$ just has the element $[e]$. Thus the definition of p_* being injective is easily satisfied. So the image of p_* is the trivial subgroup in $\pi_1(S^1)$.

Example 5.3 Let p be the covering map in Example 4.2. Then $p(z) = z^k$ for some integer k. We saw this p takes any loop of S^1 and wraps it around S^1 k times. The homotopy classes $[\gamma]$ are identified with how many times it wraps around S^1 in what direction. Therefore,the image of a generator in the covering space is $[kx]$. Thus the image of p_* in S^1 is $k\mathbb{Z} \leq \mathbb{Z}$.

Example 5.4 Let p be the covering map in Example 4.3. Although this was an interesting covering space, the induced map p_* is the trivial map even though $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$. The fundamental group of $\mathbb{R}^2$ is trivial, since any loop based at a point x_0 is homotopic to the constant x_0 path. Thus the image of p_* in $\mathbb{Z} \times \mathbb{Z}$ is the trivial subgroup.

Example 5.5 Let p be the covering map in Example 4.4. The image of p_* in $\pi_1(\mathbb{P}^2)$ is the trivial subgroup.

References

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354

[Mas77] William S. Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York-Heidelberg, 1977, Reprint of the 1967 edition, Graduate Texts in Mathematics, Vol. 56. MR 0448331